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THE MATHEMATICS TEACHER

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CONTENTS

		PAGE
Why Study Mathematics?	E. A. Bond	51
General Mathematics	Wm. S. Tobey	59
Trigonometry without Angles	C. V. Newson and John F. Randolph	66
Oversocialized Mathematics	Francis Behn Riggs	69
Pet Peeves of a Mathematics Teacher	M. H. Ahrendt	71
On Square Root by Division	E. W. Chittenden	75
The Retention of Arithmetic and Algebra in Relation to Achievement in Plane Geometry	Mildred Challman	77
A Program for the Teaching of Mathematics in High Schools and Colleges	Jennie P. Kormes	80
Models for Certain Pyramids	Joseph A. Nyberg	84
Mathematics Takes a Holiday	Wendall W. Haner	86
The Art of Teaching A Practical Project in Geometry	Fanny Springsteen	87
In Other Periodicals	Nathan Lazar	89
Program of the Annual Meeting of the National Council of Teachers of Mathematics		90
Official Ballot		92

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THE MATHEMATICS TEACHER

Volume XXXIX Number 2

Edited by William David Reeve

Why Study Mathematics?

By E. A. Bond

Western Washington College of Education, Bellingham, Wash.

Two recent experiences of mine had a part in the selection of the topic for this article. The first was an eight page editorial that appeared in a recent issue of *Life* Magazine on the theme: "War is Teaching U. S. Citizens a Belated Lesson in the Supreme Intellectual Discipline of the Science of Mathematics." Since books on mathematics are best sellers today, it is probable that teachers of mathematics should better sense the high importance of the work they are doing.

Then again in a discussion I had with one of my co-workers relating to the probable needs of China at the close of war, he challenged a statement of mine that the greatest need would be for applied mathematics. He maintained that mathematics is a tool and as such must be directed by those skilled in economics and sociology. That member of our faculty is an otherwise well informed man, but his sheer ignorance of what mathematics is led me to feel there is need for us to stress the conception of mathematics as a mode of thinking—in fact as the language in which thinking is done.

Of course mathematics has its tool phases as do all other school subjects. For example, the background of history is a tool of the sociologist and of the economist. But that mathematics, in addition to its tool aspects, is a content subject and a creative art, I purpose to show.

Now for the support of my thesis let me say it probably would have been far less bewildering for Julius Caesar to have fitted into the American way of life at the time of George Washington than it would for a Washington or a Madison to understand the way we now live and work together. The rate of change has always followed the compound interest law. One change leads to several others and the end is not yet.

One frequently hears it said that it would have been a glorious privilege to have been permitted to live at Athens in her heyday, or in Florence during the splendor of the Golden Era of the Medici, or in the brilliancy of the literary age of England during the reign of Queen Bess, or in any one of the spots that you could cover with a pin point on a world map, in which great changes were happening.[1] Yet we, who are living through changes unparalleled in world's history, very inadequately appreciate the pace, the direction, or the consequences of the changes that are now occurring in our midst.

[1] See *Progress of Arithmetic in the Last Twenty-five Years*, D. E. Smith.

Perhaps we are too close to these changes to weigh them appropriately. A hundred years later they will probably fall into perspective. Then they will be labeled for that they have accomplished. They cannot be appraised now, but in the main the changes will likely continue to be along the line of better standards of living for all people, and of more world-wide cooperation.

Aside from the war, and indeed in spite of it, this is an active period. Indeed, the changes were occurring in this country before we entered the war. Money and other wealth and their possession had ceased to have the great influence that they formerly had. In my boyhood an annuity of $20.00 per month was ample to care for one's personal needs; fifty years later $40.00 per month was needed. Now the Navy's allowance for maintenance is $81.00 per month.

On the contrary the services a man renders to others and the services he is capable of rendering or directing are fast becoming the dominant worth of a man. What he is and what he is doing are fast becoming the measure of his value to society. Now these changes have been taking place in our country and in England in rapid manner.

No man ten short years ago was wise enough to predict in the slightest degree what America is today economically, socially or politically. Something has happened and something is happening to the way of living to which the Western World has been accustomed.

Not only has our ability to make life more abundant for all mankind increased at a very encouraging rate, but at the same time and through the same agencies the ability to destroy the benefits we might enjoy has been augmented without limit, and again the end is not yet.

What has been the cause of these changes? What forces have been the instruments of these innovations? Were one to ask a dozen competent judges the answers to these questions their responses would vary somewhat, but they would present the same over-all pattern. But if our judges could be induced to list in a *one, two, three* order their conception of the agencies of the changes, among the first five of each list would be the three that follow: Language, tools, and mathematics. These are the contribution of the ages. These form the heritage which is education's business to preserve, foster, and improve. For this purpose schools are maintained.

Now I am not going to say, nor do I want anyone to think I believe, that our heritage should be passed on as it is. Indeed this heritage, precious as it is, must be continuously reorganized about our current problems, and improved to meet changed and changing conditions. As Lowell has so well said:

> New occasions teach new duties;
> Time makes ancient gods uncouth;
> We must upwards then and onwards
> Would we keep abreast of truth.

We yet need to add a fourth category to the three named above. We need to give the ethics of the Sermon on the Mount a fair trial. When we add "for keeps," the spirit implied in this sermon, we will have a better world. There are some things that are right. Laws cannot change their rectitude. Total dictators cannot stamp them out. Ultimately right will win. Total justice to all should be incorporated into our thinking. Unconditional surrender must include justice to all.

But to return to our category, I was present a few years ago when Dean Woodbridge retired from the head of the Department of Philosophy at Columbia University. He subsequently gave three lectures epitomizing his life's accumulation of experiences and observations.

Dean Woodbridge said: "Biology brought mankind up to his condition in the Stone Age. Then he was sent to school. There language, mathematics and tools were learned and his development has resulted therefrom."[2]

[2] *Teachers College Record*, Oct. 1929.

I purpose to hurriedly review with you some of the contributions that mathematics, and the sciences made possible by mathematics, has made, and the changes in emphasis that have occurred in my time. For more than thirty-five years much of my time and most of my work has been devoted to the single problem of helping to improve the teaching of arithmetic to children. There are other fields of mathematics that have undergone large improvements in the meantime and that have made large contributions to life. Indeed one of our country's leaders in the field of pure mathematics told me last year that there have been more discoveries in his field during the last quarter of a century than in all previous times, and that the Center has moved from Europe to America.

However, it has been neither my desire now my privilege to keep up with the changes in pure mathematics nor in the sciences made possible through pure mathematics. It has, on the other hand, been both my desire and my privilege to keep on the firing line and in the front line trenches with regard to the teaching of arithmetic to children.

I know my limitations and will in accordance confine myself to a topic which I know something about. I shall not be like the eminent Juvenile Court Judge who spoke on one of our College Lecture Course Programs. We had anticipated the chance to hear of his problems, and what educators could do to help solve them. Instead he gave us a discourse in Educational Philosophy. He was not through the first paragraph before each of us knew he was outside of his own experience and was talking about the schools of his boyhood instead of those of today.

But as I have implied, I have been more interested in the training of children through the medium of arithmetic than I have in mathematics *per se*. Of course a proper study of arithmetic takes one far afield. He must know how it is tied in with higher mathematics, with the sciences, and with the way we are living and working together. Mathematics is ubiquitous. It touches life in many vital spots. It should be presented to children from this point of view. Not more mathematics but more about what mathematics is like and can do is needed.

Good teaching is now and ever has been the means by which each generation is, first, made acquainted with the accumulated heritage of the race; and, second, is helped to reorganize this heritage around the current problems that each learner meets or will soon meet. Hence arithmetic is first of all a science. As other sciences are, it is an orderly classified body of facts, skills, and knowledges. It is the science of number. But if that were all arithmetic is, I would not be so ardent an exponent of its values for all children. Arithmetic is also an art. It is the art of using the science understandingly in all situations containing along with other elements those of number or measure. It is the art of giving number its just weight in making decisions relating to such situations. It is also an art of using number adequately in one's own personal affairs including his non-vocational intellectual explorations. If any subject should be the core subject in any curriculum, that subject should be arithmetic. But as yet we have not completed our definition of arithmetic. Beside its science side and its art side, arithmetic is a curriculum study. Through this phase arithmetic contributes with other subjects toward enlarging the horizons of the learners. This is so since the teacher of arithmetic is compelled to be thoroughly conversant with our economic and social undertakings and problems, and with applying number to them.

Now up to this point I could have said the same things about most of the subjects or the teaching materials for children's training. Each subject is a classified body of knowledge; each contributes to human welfare and hence is an art, and each contributes to the characteristics that mark an educated person.

But in life's situations there are many that have among other elements the element of number or measure. Often these number elements are interwoven with and obscured by other elements. So each member of the western world needs a number-mindedness sufficiently developed for him to see the number implications in the situations he meets, and to give number its just weight in the decisions the situations require. Often no computing is necessary in making such decisions.

When you hear that the population of Chicago is 3,564,000 you don't respond by adding or subtracting. Perhaps you sense it would make 100 cities of the size of Bellingham, but in the main you would respond by some comparison such as this. More than half the people of Illinois live in Chicago. Again when you hear that the Government wants a loan of $14,000,000,000 you don't add, subtract, multiply or divide. You might try to gain some conception of the size of this number. You might sense that the number of minutes since the birth of Christ is about a billion, and so the number of minutes in the next 28,000 years is about 14 billion. So if this amount is repaid at the rate of $100 per minute it would take 280 years to repay this $14,000,000,000. Or you might think, "It would take a billion silver dollars laid side by side to cover the equator. Hence a belt 14 dollars wide all round the earth would be 14 billion dollars." This would be 10,800 car loads of silver, or 200 train loads of 50 cars each. No, the responses to number are more usually than not responses other than figuring.

Now the science of arithmetic has not changed much in my time. Indeed a textbook of my boyhood would provide a good vehicle for teaching the science today, but the emphasis and the uses have changed very much. They are in continuous flux. The uses that give point to the teaching of the science and that provide the illustrative material which gives meaning to the science are changing all the while. So does the psychological order of presenting the different phases of each topic change rapidly. It would be a grave mistake to neglect the science for the art. We need both values. Moreover, since the correlation between their learning is not very high, each phase will need attention. One cannot use tools he does not have, on the one hand, and one must learn how to use the tools that he does have on the other. The amount of transfer is given by the equation: $T = 1 - \sqrt{1 - r^2}$ and $r = .50$ which means that only about 13% of the energy spent on one transfers to the other. One cannot learn to figure by solving problems, and one cannot learn to solve problems by learning to figure. So both phases of teaching number are needed.

When I spoke a little ago about the changes in the social uses of arithmetic, I was speaking only of the need for continuously scrutinizing the illustrative material which makes arithmetic meaningful, for the purpose of assuring ourselves we are using the material which will prove serviceable to children now and later when they take their place in the economic world.

Let me illustrate. One summer vacation when I was a student in college, I went over to the wheat fields of eastern Oregon to work. The crew with whom I worked cut and threshed more than 50,000 sacks of wheat on one farm. After the harvest was completed I was one of the helpers who were retained to help haul the wheat to market. I was given eight horses strung out two abreast. They were hitched to two wagons, coupled up tandem fashion, and loaded with ninety sacks of wheat. It was my job to herd that eight-horse team down the 3,000 foot grade to the Columbia River, unload the wheat, camp on the sand at the river, and drive back the next day with part of a load of wood. It took two days to get ninety sacks of wheat to market. Even green hands eventually learned to hold four lines in each hand and transfer the eight into one hand while he took appropriate measures to see that each horse did its part.

A year ago last summer when returning from San Francisco I passed the farm where I formerly worked. On the highway there were trucks loaded with 200 sacks of wheat making four round trips a day. It is no longer necessary to teach young men to drive eight-horse teams. Hence such should not be a part of a college program.

It is hard to keep tradition from playing too much influence in what we teach, e.g. in relation to our measures and common fractions. In my time there have been many changes in our way of teaching arithmetic. Some of them have made valuable contributions toward improvement. Others have had to be lived down as have other fads in education. Let me recount some of these changes which I deem most important. Then I shall set forth our present status. I have been a member of the last two National Committees on the Teaching of Arithmetic appointed jointly by the National Education Association and the National Council of Teachers of Mathematics. As a member I have attended committee meetings in all the principal cities of the nation, twelve meetings in all. Accordingly, I feel pretty close to the combined judgments of teachers of mathematics of the Nation. Accordingly, I can speak from the authority of the committee.

At about the turn of the century curriculum makers began to be skeptical of the amount of arithmetic that was taught. They questioned the usefulness of some topics that were being taught. A committee of fifteen was appointed by the N. E. A. to study the situation. This committee reported that many techniques had been extended far beyond their needs, that the work should be simplified to accord with usage, and that the time set apart for arithmetic should be reduced. At the same time many other areas of instruction were being added to the school subjects; art, music, health, elementary science, manual training, household arts, etc., were among them.

As a consequence of both these factors, the time for arithmetic was reduced from 19.2% of the total school time for the eight grades to 10.0%. There was a decline in both the quantity and quality of arithmetic instruction. The time is now increasing slightly and is 10.8% at present. At that time the slogan was: make arithmetic *practical* (whatever that means). If you will let me define *practical* I will agree with that *thesis* today, but the most practical kind of arithmetic is that which ultimately equips a learner with a unified science of number—a number-mindedness—that is adequate to interpret numbers read, heard or thought. But to be interpreted narrowly to mean: bread, butter and potatoes I must vote against it.

By way of illustration it is far more important for most people to be able to know when to borrow money and when to lend it than it is to be able to compute the interest upon a note. Far too many people yet buy endowment life insurance contracts when their sole need is for protection, and far too many people live upon their next month's pay check instead of their last month's salary. Far too many people are caught by the easy payment plans to the extent that they budget their expenses far above their income. In many other ways the lack of number sensitiveness is a serious menace to a family's advance toward financial well-being.

The next change I shall mention took place in the teens of the century. There came the testing movement. Measures and facts began to replace theory and opinion in many phases of education, but since it was far easier to develop tests that measure the mechanics of arithmetic than the other and far more important value, computational tests became far more numerous than measures of the other values. As a consequence teachers attempted to prepare their pupils to do the mechanics of arithmetic to the exclusion of the other values. We then had what was called the drill theory of the teaching of arithmetic. For more than a decade it was attempted

by sheer drill to beget habits of accuracy and reasonable speed but learning without understanding has always been short-lived. The tests revealed that even though 95% of the time spent upon arithmetic was devoted to attempts to get precision in adding, subtracting, etc., there was no improvement. Moreover there was a decided loss in the ability to apply numbers to life's problems.

Two years ago I had some skilled research workers compare the responses of our entering freshmen for the year 1939 with those of the group that entered in 1928. The same form of the tests were used in each year and the aptitude rating was the same for each group. In every way the former group surpassed the latter in both computational arithmetic and in arithmetical reasoning. Drill without meaning is uneconomical.

As a revolt against the inadequacy of drill there developed the fad that number learning was started too early, so in the lower grades incidental learning of numbers was advocated. It sounded well, but the only trouble was that it wouldn't work. There was a great variation in the amount and quality of instruction, ranging all the way from little or none to a rich program of number experience. For too many teachers, such a program became both accidental and casual, while in even the most favorable conditions the instruction became haphazard and piecemeal. It could not result in the orderly learning of number. It could not culminate in number-mindedness or number-sensitiveness.

So late in the twenties the meaning theory became uppermost. It was felt children have a right to see sense in what they learn. Reasons for learning, and the significance of what is learned should precede practice. Insight into the relation of what is being learned to previous learning was felt to be desirable for economic learning. This is the status today: *Meanings, significance, insight.*[3]

Any new topic of arithmetic should be shown to grow out of some need for the topic. The relations of this topic to previous learning should be evident. Other needs of the topic should be canvassed. Then sufficient practices should be given to insure the learner's ability to use the topic intelligently whenever and wherever such ability is needed. Arithmetical skill without meaning and significance is not mathematics. Mathematics requires reasoning—the process of selective thinking that leads to proof. This is the supreme privilege and exercise of human intellect. Mathematics requires no numerals and little figuring. Its conclusions, the only absolute truths we know, are expressed as equations couched in general number based upon nine absurdly simple statements or assumptions all which tally with experience. From these few assumptions multitudinous other truths are deduced by pure logic. The process of such deductions and the results are what we know as mathematics. Unfortunately, there is too much computing and too little mathematics taught in our arithmetic classes. Hence arithmetic is both the beginning and the end of mathematics for the vast majority of people. One reason for the decline in the amount of mathematics taught in our schools is the feeling that it is too hard for many children. Of course we cannot teach to all the children the intensive type of mathematics that was formerly taught to a much smaller group of highly selected pupils. But it seems evident that we both can and should give as much insight as possible to all students into the language of algebra which is the language in which the thinking of the world is done. At the same time our more capable students should not be limited by the abilities of the less able.

There has been an age old conflict between the pure mathematicians and those who feel that all mathematics should be applied mathematics. The only trouble is that we cannot draw a line and say all on

[3] See Buckingham, R. B., THE MATHEMATICS TEACHER, Jan. 1938.

one side is pure mathematics and all on the other is applied, since it often happens that a discovery in pure mathematics finds an immediate use in our economy. A Newton develops the binomial formula by pure thinking; a Mendel uses it in biology, and a Pearson in statistical analysis. A Helmholtz studies the theory of recurring imaginary number and an Edison uses it to hitch together alternating currents to make a continuous one. A Michelson studies the rate of travel of ultra violet rays during a total eclipse of the sun, and an Einstein uses his research to expound some needed changes in Newton's laws of motion. A Taylor studies the interference of returning waves of the sine curve, and we sink Japanese ships that we have never seen. Incidentally, the Japs went beyond this. They sank our ships before they were launched.

A student plotting the graphs of the moons of Jupiter asks an astronomer to point his telescope at a particular spot in the celestial sphere and the latter discovers a planet not previously known. And just within the last few days the work of the professional gambler Cardan, developed wholly as an aid to those bucking a game of chance, has found a use. This work is being used by the synthetic chemist to find how many different combinations of the elements in glycerine are possible in order to increase the explosive force of high explosives. No, you cannot separate pure mathematics from applied. My guess is that the island of known is very small as compared with the vast expanse of the unknown. Now there are two types of scholars. One class digs at the margin—each in his own particular field. The other climbs an elevation overlooking as much of the discovered truth as is within the horizon of his ability. No one can say one of these types of scholar is more valuable than the other. We need them both. One Helmholtz will make an Edison in every generation.

In support of my thesis, the following quotations are pertinent:[4]

1. Mathematics is the Queen of the Sciences. Gauss
2. The Science of Mathematics may claim to be the most original cration of the human spirit. Whitehead
3. The power of mathematics rests on its evasion of unnecessary thought and on the wonderful saving of mental operations. Ernest Mach
4. The mathematician reaches truths of endless importance for our description of the physical universe. Karl Pearson —Biologist
5. The new physics, if not purely descriptive of experimental work, must be essentially mathematical. Dirac
6. Mathematics offers the Natural Sciences a certain measure of security which without mathematics they could not obtain. Albert Einstein
7. To create a healthy philosophy you should renounce *metaphysics* and be a good mathematician. Bertrand Russell
8. How can it be that mathematics, being after all the product of the human mind, independent of experiences, is so admirably adapted to objects of reality? Einstein
9. Every new body of discovery is mathematical in form because there is no other guidance we can have. Charles Darrow
10. Mathematics is the most exact of sciences and its conclusions are capable of exact proof. Steinmetz
11. Mathematics is the only good metaphysics. Lord Kelvin
12. Algebra is the language of thinking. Whenever and wherever there is any thinking algebra is used. Irving Fisher
13. Algebra won't open all doors, but it will get you places you couldn't go without it. Hotelling
14. Arithmetic is the most intellectual game that has ever been devised. Thorndike
15. No man is the same after he first meets the algebraic equation. Charles Judd

[4] See Erick Bell, *Men of Mathematics*, pp. xix ff.

The Gist of What Has Been Said

The world has real need for mathematics. It has been invented to satisfy this need. Our way of living could not go on without it. Life presents many situations that demand quantitative solutions. In the beginnings these solutions have been vague and formless as is the case in economics today. Gradually, however, these solutions are refined and perfected as has happened in our measure of time and of energy changes. Mathematics has persisted and will persist because it satisfies these needs. It does so by precisely measuring amounts and by forming equations showing the relations of changes in the amounts. Then these equations of relations give a means of control and the possibility of predictions of the results of changes in the amounts. In this way the mathematical science has been developed.

Pure mathematics long has been an enticing field. Man has ever marveled at its many novelties. Perhaps too many mathematical research workers have been carried too far afield by the nicety of its processes and the precision of its results. As a consequence both the processes and the extent of precision have gone beyond their present uses. On the other hand and at the same time there has been a clamor for popularizing mathematics by diluting it to the point where there is neither rigor nor precision. Both of these tendencies have affected adversely the mathematics that is taught. As examples of the latter tendency we note many books on "The Theory of Statistics," on "Navigation" and on "Physics" which assure their readers they "need no mathematics beyond the rudiments." But unless publications in any such fields use mathematics much beyond the "rudiments" they are either elementary or fragmentary or both. Their contents have been built from the top downward, rather than up from the foundation. Any serious study of any of the arts or sciences based upon measurements requires a type of mathematics which is above the level of undergraduate schools.

It is neither of the two extremes that has caused mathematics to persist. The type the world needs is that which helps it to control the conditions it meets by precisely measuring amounts, by showing the relations among these amounts and thus giving the basis for predictions and the means of controlling these amounts. This is the mathematics needed by the world. This is the mathematics that is being improved very rapidly. This is the mathematics without which our world would begin to disintegrate within a single generation. But with which it will go on improving the conditions it meets because it can control them.

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General Mathematics

By WILLIAM S. TOBEY
Junior and Senior High Schools, Long Branch, N. J.

DURING the past four or five years volumes have been written on the reasons for adding general mathematics to the curriculum. Much has been written in general terms on the content of such courses. We are making little progress. We have had years of war with its emphasis on the need for mathematics but relatively little of a fundamental nature has been accomplished. There appears to be a lack of understanding of what needs to be done and how to do it. We have theorized too long. We set up goals and objectives, and then more goals, but just what to do, and how to go about doing it is seldom treated. It is time, yes past time, that we start doing something. Even though we do poorly and make many and serious mistakes we will have created possibilities for correction and improvement. All things must have beginnings and only after a thing exists can it be modified and improved. In our system we began some seventeen years ago to blunder along in an uncharted field, toiling in the belief that with constant analysis of our work we could improve.

General mathematics for the present and probably for some time to come must be almost exclusively remedial. We must re-teach nearly all the concepts that were supposedly taught in the earlier grades. Most that these pupils bring with them to the ninth grade is a mass of misunderstood or only partially mastered techniques for manipulating.

One of our first tasks is to build meaning into subtraction when borrowing is involved. This problem is complicated to some extent by there having been two methods taught earlier. About half of our pupils have been taught to increase the number in the subtrahend while most of the other half have been taught to decrease the minuend. Fortunately for our work, these pupils have had so little experience in dealing with denominate numbers that we have here what is practically a new and interesting field. This is very important with pupils of this type. They do not want drill and will profit little from it if it is required of them. When first confronted by a situation where yards, feet, and inches are to be subtracted from units of the same kind they will frequently change all to inches. Once they have discovered that it is much easier to borrow when necessary they accept willingly and will work a large number of exercises with apparent interest. After a day of this we are ready for subtraction of a more abstract kind. We go directly to subtraction of two or three digit numbers. In case the exercise is to subtract twenty-seven from forty-two the first is written as two tens and seven ones, and the second as four tens and two ones. We point out to them that just as they borrowed one yard and changed it to three feet so here they borrow one ten and change it to ten ones. If some of us feel that this is too simple let us ask pupils in any of our high school mathematics classes why it is permissible to borrow. If it is truly clear in the minds of our writers of textbooks and treatises on the teaching of arithmetic it is difficult to understand why so many pupils are taught to increase the number in the subtrahend.

This first illustration of our method may have raised questions regarding the theory upon which our procedure is based, the method of motivation, and goals. We believe the program of the early grades is relatively good except that the element of understanding is too frequently omitted. We believe that much prior learning is in a dormant stage and will take on vitality once understanding is injected. To motivate interest, our pupils are told in a

matter of fact manner that their progress in arithmetic has not been good, and that this is due primarily to their not having understood the topics as they went along. We remind them, that, like most boys and girls, they liked arithmetic when quite small. We ask them to recall that it was only after they were required to learn a lot of rules, and work exercises without knowing exactly what they were doing, that they began to lose interest and in some cases to dislike arithmetic. We promise them that if they will do their best, and always insist on knowing the "why" of everything that is done in class, they will make progress and enjoy their work. Pupils are at times invited to bring problems to class but not too much stress is placed on this aspect. To teach effectively requires organization. The teacher must lead to the proper activities. Incidental learning has no prominent place in our system. The goals are to make the arithmetic of the first eight grades meaningful, enrich it where possible, build confidence on the part of the pupil, and develop an interest in mathematics to the end that these pupils will elect general mathematics or, in a few instances, sequential mathematics in the tenth, eleventh, and twelfth grades.

We treat division in much the same manner as subtraction. Here is one of the finest opportunities to build enthusiasm in conjunction with understanding. Approximately ninety per cent of our pupils change all quantities to the lowest denomination before dividing. Once they are shown that it is much easier to change only remainders to the next lower unit, they will accept drill with apparent enjoyment. We always conclude this phase of division with exercises involving dollars, dimes, and cents. They are surprised to learn that the procedure they use with these units is identical to the one they always use when dividing one number by another. This frequent throwing of light on topics, which have hitherto constituted totally dark spots, is a powerful factor in maintaining interest. It is probably more effective than the emphasis on job problems, play problems, and pictures so frequently advocated by many who have not actually determined the relative values by experimentation. At the end of two class periods of this kind of activity, pupils of this age are ready to attack the more abstract exercises. Only relatively simple cases are attempted since our aim is understanding. There must be no confusion. At first a number such as 348 is divided by 4. The three hundreds cannot be divided by four so we must make some change. Referring to the work of the day before, where such a quantity as three yards, one foot, and four inches was divided by four, we come to the conclusion that the three hundreds and four tens must be treated as 34 tens. Just as dividing thirty-four dollars among four people would give each person eight dollars with a remainder of two dollars, so dividing thirty-four tens by four yields eight tens with a remainder of two tens. Since our quotient so far is in terms of tens we will, for the sake of orderliness, write it above the four of the dividend. The remaining tens can be changed to twenty ones and added to the eight ones just as the remaining feet in the problem the day before were changed to inches. We now have twenty-eight ones which when divided by four yield seven ones. A number of problems involving simple division are introduced at this stage.

In solving problems, an estimate of the answer always precedes the actual solution. Pupils at this age enjoy approximating. They probably feel somewhat honored by being expected to use judgment. To be asked what they think about a situation is something of a novelty to these pupils, accustomed as they are to being told that this is the way to do a thing and directed to do likewise. A little guidance quickly brings about a tremendous gain in the ability to approximate. This adds greatly to their self confidence. The joy which is theirs, when they learn that there is to be no learning of rules, adds to the zest with

which they accept new problems. In this work judgments are always based on what we have previously discovered. Pupils like to expound the results of their experimentation; they hate to memorize rules.

A great vacuum in the earlier experiences is due to a lack of any real notion of length, area, and volume. On the ninth-grade level, we treat them as one unit. We start by building inch cubes. Frequently each pupil must build six of them since fewer than 144 would be of little value. Once, however, they really see that it takes 144 of them to cover one square foot, they can be convinced by lengthy discussion and much figuring, that to build a block one foot long, one foot wide, and one foot high would require 1728. This is an enormous number. Probably few adults really believe it to be true. Much time and effort are expended on this unit. Boxes of many sizes and shapes are compared to the little cube. A box six inches by five inches by four inches is placed on the teacher's desk, the little cube is set beside it, and members of the group estimate the number of small cubes needed to fill it. A member of the group is chosen to place the cubes on the box. When every member is convinced that thirty are required to cover the box the teacher places four of the cubes one upon another so that all can see that four layers are needed to fill it. The act of estimating brings with it a considerable amount of the competitive, and this, probably more than any other element, raises the interest to the extent that the long treatment necessary to mastery is possible. Teachers who have taught solid geometry know that even our strongest pupils in the twelfth year of sequential mathematics have only a vague concept of volume. Estimating and computing the dimensions of rectangular containers equal in volume to a gallon, a bushel, or a ton of coal are interesting exercises. Drawing figures with labeled dimensions helps to fix the idea of volume and its relation to surface measure and length. Cans can be brought to class and their volume checked.

This work on volumes invariably leads to questions regarding the volumes of cylindrical solids such as tin cans. This opens up a grand field for exploration. First to be met is the demon *pi*. The notions of *pi* held by pupils are truly fantastic. Every pupil is required to measure the circumference and diameter of a cylinder. Each is required to divide his measure of circumference by that of the diameter carrying his result to the nearest tenth and no more. The results of the measurements are averaged and if this is in error in the hundredths place the experiment is repeated. Our goal is 3.14. Seldom are more than two attempts necessary. An analysis of this project shows that an enormous amount of arithmetic is involved. This is the type of review and drill that pupils of this age will accept. Since few of the group have ever used rules marked off in tenths, they are permitted to use the kind with which they are familiar. Their divisions are written in the fraction form, both members multiplied by the common denominator, the integral part of the quotient removed by inspection which in every case consists of subtracting three times the denominator from the numerator, and changing the remaining fraction to tenths without recourse to division. In this grade no opportunity to destroy a false notion that has crept into their thinking should be lost. They come to us with the belief that data of a low order of precision can be made to yield an exact result merely by applying extended accurate computation. For example, a pupil must be made to realize that if his measurement is, let us say, three and thirteen sixteenths it is, when changed to the decimal form, 3.8 and not 3.81 or 3.813. We must be patient with our pupils at this point for we are attempting to destroy one of their cherished beliefs, and we are, to them, decidedly radical. Everyone that they know does it the other way. The meaning and significance of an *average* is being built. Pupils are surprised and their self confidence strengthened when they discover

that their own little group, by working cooperatively with relatively crude instruments, can find a value of *pi* just like the one in the book. Now it is their *pi*. After the battle has been won the pupils are given cross section paper marked off in tenths and the experiment is repeated. New tools add zest to an endeavor.

Pi r^2 and 2 *pi r* are certain to appear at this stage. A little questioning will prove to anyone that they have no clear idea of their meaning or their use. Some will insist that the former gives the area of a circle while others will hold that the latter is the formula for determining the area. They are asked to be patient for a few days while we do a little experimenting. We draw six-inch circles inside six-inch squares on quarter-inch cross section paper and have the pupils count the small squares inside the circle as their homework assignment. The following day the results are averaged and compared with the number of small squares in the circumscribed square. Some pupils have discovered methods of constructing squares or rectangles within the circle as means of making the counting easier. Much is made of this and the teacher makes other suggestions so that before the end of the period the necessary effort is so reduced that the pupils are anxious to try again. Occasionally two or three attempts must be made to arrive at the required .785. Once this has been achieved we set out to find a simple fraction which has very nearly the same value. Throughout a large part of their school experience these pupils have had much drill on changing common fractions to decimals but here is a novel experience. The possibilities for practice in estimating, approximating, comparing and making proportionate changes in the members of a fraction are great. This is teaching pupils to think quantitatively. Few adults know how to find the area of a circle. Pupils who had this experience as long ago as fifteen years respond unhesitatingly that the circle fills about eleven-fourteenths of the square. We do not forget quickly the things that we have truly understood. *Pi* r^2 has never been understood by any large part of our school population. All that is being said about the retention and usefulness of mathematics is scarcely worth more than the most casual consideration due to this defect in our teaching. We must have intense interest over a relatively long period in order to utilize the laws of learning but this does not mean as appears to be implied in much current discussion, that we must make frequent trips to the woods, fields and factories. Mathematics has within itself the most intriguing goals.

We now have a connection between the circle and the rectangle and are ready to begin the study of the volumes of tin cans. An analysis of the arithmetic involved in checking the contents of a can of soup belies the need for routine drill. We discover that the diameter of a Campbell's Tomato Soup can is a little more than $2\frac{1}{2}$ inches and the length about $3\frac{3}{4}$ inches. If it had a square base its content would be $2\frac{1}{2}\times2\frac{1}{2}\times3\frac{3}{4}$, but we must find 11/14 of this since the cylinder fills that part of the rectangular solid. The pupils must multiply four fractions and, of course, change the resulting mixed number to a form compatible with their original measurements. Many of the pupils will arrive at a result of 18 93/224. Nothing in their measurements warrants any such denominator. It must be changed to sixteenths. To teach effectively at this point the teacher must coax and prod the group until someone discovers that the 224 and 93 must be divided by 14 to change the fraction to the proper form. Since 14 is contained in 93 more nearly 7 times than 6, the result must be written as 18 7/16. Others will obtain an answer in the form of a whole number and decimal. In this form the result must be written as 18.4 and not as 18.41 or 18.415. Since a gallon contains 231 cubic inches, the number 18.4 must be divided by 231 and this result multiplied by 128, the number of ounces in a gallon. At this point the pupils

are required to pause in their computations to estimate. They see quite readily that 18 is a little more than $\frac{3}{4}$ of 23 so the first digit in their quotient is probably 8. But 8 what? This is not easy so we write the 18 over 231 and ask for estimates of the value of the 18 when the 231 is reduced to 100. All will see that it would be less than 9, so our 8 arrived at a few minutes earlier must be .08. The content of our can is about .08 of a gallon or .08 of 128 ounces. Multiplying .08 by 100 would yield 8; 28 is a little more than a quarter of a hundred so we must add a little more than one fourth of 8 to the 8. Our result is slightly more than 10 ounces. We continue this reasoning until every member of the group is satisfied that there are about ten ounces in the can. After that they are permitted to complete the computations. They arrive at a result 10.2 and the company states that the content is $10\frac{1}{2}$ ounces. We recall that the diameter is a little more than $2\frac{1}{2}$ inches so probably the company is honest. Frequently the interest of ninth-grade pupils can be held on this topic for more than a week. During this period they are receiving a rather intense drill on multiplication of fractions, division and estimating in a form which they do not resent.

Once the pupils have acquired a good understanding of the relation between the area inside a circle, and that of the circumscribed square, the time has arrived to take the mystery out of $pi\ r^2$. A circle with its square is drawn and two perpendicular diameters which are also perpendicular to adjacent sides of the square are added. The pupils readily see that the larger square is divided into four smaller squares and that the side of the smaller squares is the same as the radius of the circle. The area of each small square is r^2. From this they discover that the area of the larger square is four r^2, and the area of the circle must be 11/14 of this quantity or 44/14 r^2. This reduces to that familiar 3 1/7 that they have heard so much about, and about which they knew next to nothing. After this exercise we solve a number of problems involving circles by first treating them as 11/14 of the circumscribed square and then by the formula, A equals $pi\ r^2$. By this method of treating the formula, a two-fold purpose has been served. Its introduction has increased the interest in solving problems, and solving problems has intensified the interest in the formula itself and extended the period of useful drill.

We treat areas of triangles, parallelograms, and other straight line figures by the scissors method. There are those in our profession who believe that ninth-grade pupils have outgrown this method. Actual experience has proved that the belief is false. Opportunities to actually see truths have been so rare in the lives of these youngsters that they truly delight in determining the area of a triangle by first changing it to a rectangle and then comparing it with the unit square. Discovering that two quite different rectangles can be formed from the same triangle adds decidedly to their interest in the project. It will be noted that in our plan the area of the triangle does not follow directly that of the square and rectangle. There are three reasons for this procedure. At this stage of their development interest is more effectively retained by moving immediately to volumes. Probably both area and volume are more easily grasped when treated together, and additional opportunities are provided for reviewing the units of measure. Spacing and repetition receive too little consideration in much of our teaching.

The volumes of pyramids and cones are discovered by actually comparing them with prisms and cylinders. Pupils construct cones and cylinders of the same diameter and altitude, and compare their volumes by filling the former with sawdust and pouring it into the latter.

In order to build meaning into fractions, frequent short drill periods are devoted to solving simple fractional equations without pencil and paper. We ask them to solve such an equation as 5 over 8 equals some-

thing over 10 by noting that 10 is two greater than 8, which means that it is one-fourth again as large as 8, so the numerator must be one-fourth again as large as 5. The 5 must be changed to $6\frac{1}{4}$. Of course we do not call them drill periods but periods when we treat fractions by the brain method. Pupils of rearly equal abilities must be grouped when the work is done at the board, and when by responding to a partial equation written on the board by the teacher, care must be exercised in choosing the example to fit the pupil. Denominators of 10 and 100 occur frequently in these exercises since these lead the way to decimals. Decimals are treated as special cases of common fractions. Because of the peculiar form of our number system it is possible to write fractions without denominators in case the denominator is 10, 100 or any other number consisting of 1 and zeros. This leads to discussion of the place value of digits which in turn dilutes to some degree the mystery surrounding the whole matter. If the 2 in 82 is worth a nickel, what is the value of the 8, is a type of question that pupils appear to like. If they find it too difficult it can be changed to, if two quarts of milk are worth 36 cents, what is the value of 4 gallons of milk? The teacher must not permit the pupils to find the price of one quart and then multiply by the number of quarts in 4 gallons. They must be required to reason that since a gallon is four times as large as a quart, and since there are twice as many of them, the value must be eight times as great. There are so many instances in the teaching of mathematics where the teacher must have direct control of the method of reasoning, that homework must be assigned with extreme care, if not decidedly limited. We must keep in mind that our generation was brought up on the unit analysis method, that mother, father and friends do help pupils with their homework, and that by the jungle method of waving hands and calling on the pupil most in evidence, we are merely checking on the best neighborhood brain with not the slightest clew as to the state of mind of the pupils. This kind of thing goes on every day in every part of the country.

Whenever there is occasion, or the occasion can be created, to change a common fraction to the decimal form, we require the pupil to multiply both numerator and denominator by a number that will make the denominator 10 or 100 or as close to these numbers as can be done readily. They are then required to raise or lower the denominator to 10 or 100 and change the numerator proportionately. In treating a fraction such as 22/83 the pupils are trained to note that if 17 were added to the 83 it would become 100. Since 17 is a little more than one-fifth of 83, we must add a little more than one-fifth of 22 to the 22. Some will suggest that we add 4, others that we add 5. Occasionally $5\frac{1}{2}$ is suggested. Interest has been aroused and will linger long enough to draw considerable thinking from the pupils. Thinking is our goal. Through it the brain may develop. Mere finding of answers, if the results of the last twenty years are criteria, does not serve our purpose well. To require pupils to change common fractions to decimals by dividing the numerator by the denominator is dull drill; to check the result of their approximation by this method is to them something entirely different.

The fraction concepts are disastrously incomplete. So early were these pupils taught the mechanics of multiplying numerator by numerator, and denominator by denominator, and inverting in the case of division, that the ratio and division concepts are non-functional. Two-thirds is one-third of two and when used as a divisor yields a quotient three times that obtained were two so used. This is new understanding to these pupils and, if skillfully handled, can be made to produce a large amount of mental activity. That the word reciprocal is a tongue twister is unfortunate but why it could not be eliminated without discarding the idea is unanswerable, yet it apparently has been

done. This concept is quite well developed when the child enters school and could be built up continuously. It must be built in order to clear up the confusion now enveloping division of fractions. We should do it in our ninth-grade general mathematics classes.

One often hears voices of despair when the matter of problem solving is mentioned. Many hold that these pupils cannot be taught to solve problems; few in our sequential courses shine at this task. Our pupils have never been taught how to solve problems; they have been given formulas or patterns to follow. They must be taught to read. We must refrain from letting the bright pupils answer all the questions, telling them, and accepting mere guesses. They must be questioned and their responses questioned until they become critical of their thinking. We require them to analyze a few problems during class periods rather than showing them how to do a certain type and then assign many more for them to take home to their parents and friends. Only the classroom teacher will extract the solutions from the pupils and only by this process does the pupil gain in ability. This may seem like an impossible undertaking, but it can be done to some degree, and must be done, for the goal of mathematics is to equip a generation to solve its problems. If we fail in this aspect, we fail completely.

The situation as we face it today appears about as follows. Our civilization needs more mathematical training than now exists. This deficiency is most acute in the elementary field. Our training in the lower grades is not adequate and despite the great effort now being made by groups in our state, the results of their labor cannot have a marked effect for a long time. It probably can never raise the standard to a height consistent with the complexity of our age. We can afford additional training in mathematics for all youth. Those pupils who make up the group sometimes labeled "slow-moving" do want to learn. They can learn. Rarely have they failed to make gains. The average gain in one school over a period of several years was approximately 150 per cent of normal. They are no more difficult to teach than pupils of the sequential classes, and are no less cooperative and appreciative. To be sure, the teacher needs special training for this task, but any good and sympathetic teacher can make rapid growth in this endeavor. General mathematics has been taught for nearly twenty years. The special disabilities of this group are quite well known and the techniques for meeting them rather well developed. Plenty of good content is available for four years of the subject and at least one school in the state offers it for that period. All that is needed is courage to go forward with the project, and it is my firm belief that the end of the next decade will see four years of general mathematics offered in practically every school of the nation.

To a Snowflake

By JOAN HOLLANDER, *Columbia High School, Maplewood, N. J.*

Oh hexagonal, triangular, colorless creature,
Essence of geometry in every feature.
Your contours alone are glorified
When your angular symmetry's magnified.
For you the jeweler or baker cares,
Your design he copies for his wares.
Along with the triangle, rectangle, and prism,
Geometry books should teach "snowflakism."
With sisters and brothers of similar form,
You make the setting of a true winter's morn.
Highest of praises to you may be sung,
Without you "White Christmas" could never have come.

Trigonometry without Angles

By C. V. Newson and John F. Randolph
Oberlin College, Oberlin, Ohio

Either by definition or by implication, the great majority of textbooks on trigonometry regard the subject as fundamentally concerned with angles. In fact, the idea is presented with such insistence and emphasis that advanced instruction in science and mathematics is definitely handicapped. Whenever the expert on wave theory writes a collection of trigonometric functions upon the blackboard, he must expect the question, "Where are the angles?" The physicist explaining the use of sine functions in the study of vibratory phenomena invariably discovers that his students are puzzled by his failure to introduce angles somewhere in the discussion. In calculus the trigonometric transformations are a perpetual headache for students who have been disciplined to always think of cos x as meaning the cosine of *angle* x. Very good scientists have confided to the authors of this article that the graphical representation of $y=\sin x$ upon a cartesian axis-system always bothered them because they had difficulty accepting the device of marking x upon a straight axis; such a situation undoubtedly resulted from the false emphasis found in our textbooks on trigonometry.

To define trigonometry in terms of angles is to base one of our most important mathematical systems on one of its applications. Obviously, an intolerable condition results. It is true that the trigonometric functions are frequently applied to problems involving angles, as in surveying; this may occur about 5% of the time; but in the other 95% of the cases, the argument x in sin x, cos x, or tan x may denote time or some other magnitude, or just a number without any other connotation. In view of these facts, a definition of trigonometry in terms of angles is about as reasonable as an attempt to define arithmetic as the science of money, since perhaps 5% of the applications of arithmetic are to be found in problems involving the making of change, the figuring of taxes, and so on.

Trigonometry is said to have originated with Hipparchus (about 140 B.C.) as he considered certain arcs and related chords involved in his study of astronomical triangles. The early Greek and Hindu works on trigonometry were always concerned with the trigonometric functions of arcs, not angles. In fact, it was not until 1551 that the first book appeared in which the six trigonometric functions were treated as functions of an angle, and essentially as ratios; this was the *Canon doctrinae triangulorum* of Rhaeticus. Trigonometry as a modern mathematical science had its origin in the seventeenth century, after the development of a satisfactory symbolism. Probably the greatest contribution to the basic theory of the subject during that period was the derivation by Sir Isaac Newton of the power series representing sin x and cos x; they are

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots + \frac{(-1)^{n+1}x^{2n-1}}{\lfloor 2n-1} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots + \frac{(-1)^{n+1}x^{2n-2}}{\lfloor 2n-2} + \cdots.$$

These series appear in Newton's *De Analysi per Aequationes Numero Terminorum Infinitas*, which was written sometime before 1669, but was not published until 1711. It is common at the present time to develop these series by methods of the calculus, but more elementary derivations are possible. Note, for example, the methods recently employed by J. V. Uspensky,

(Math. Notae 4, 1–10, 1944).

In developing the power series for $\sin x$, Newton employed term by term integration upon the power series representing $1/\sqrt{1-x^2}$ in the vicinity of the origin, thereby obtaining a representation of arc $\sin x$ in the form of a power series. By a process of inversion, he then deduced his series for $\sin x$. It is not clear from Newton's work whether he realized the importance of his contribution; he appeared to regard the derivation merely as an example of some possible manipulations upon infinite series. Certainly, however, he had no thought of restricting the x in $\sin x$ to angles. In fact, no analytical studies of the trigonometric functions require the restriction that the arguments be angles.

A general definition of trigonometry acceptable today would simply say that trigonometry is the mathematical science concerned with the trigonometric functions. The trigonometric functions are the sine, the cosine, and their ratios and reciprocals. Basically, $\sin x$ and $\cos x$ could be defined by their series, wherein x may be *any finite number*. The mathematical system introduced in this way is valid when x is either real or complex, and all the usual trigonometric formulas and principles may be derived without reference to any geometric construction. The reader unfamiliar with this type of development may refer to Whittaker and Watson, *A Course of Modern Analysis* (American Edition, 1944), pages 584–588.

It is readily admitted that the development of trigonometry suggested in the previous paragraph is not advisable in an introductory course. If, however, x is a real number, there are other devices for defining $\sin x$ and $\cos x$. Possibly the most elementary process involves a unit circle with its center located at the origin of a rectangular coordinate system. It should be understood that the introduction of a coordinate system at this point is merely an expedient for displaying the correspondences between a number x and the numbers $\sin x$ and $\cos x$. The particular form of the coordinate system thus employed in formulating the definitions of $\sin x$ and $\cos x$ should not be confused with that used later in graphing such relations as $y=\sin x$ and $y=\cos x$.

To be more specific and to explain briefly this type of development, consider the reference circle of radius 1 with its center at the origin of a rectangular coordinate system. Let x be a real number, positive, negative or zero. Start at the point (1, 0), and lay off on the circle an arc whose length is the numerical value of x, measuring the arc in the counterclockwise direction if x is positive, but in the clockwise direction if x is negative. Assign the coordinates $(\cos x, \sin x)$ to the other extremity of the arc, thereby defining $\sin x$ and $\cos x$. Finally, define the other trigonometric functions in terms of the ratios and reciprocals of the sine and cosine functions. As just described, $\sin x$ and $\cos x$ are numbers whenever x is a number. It is therefore quite natural in the later development of the subject to represent the number-number relations $y=\sin x$ and $y=\cos x$ on an ordinary rectangular coordinate system.

It is immediately apparent from the definitions that $\cos 0=1$ and $\sin 0=0$. Moreover, for each number x it is readily shown that $\sin(-x)=-\sin x$, $\cos(-x)=\cos x$, and $\sin^2 x+\cos^2 x=1$. Also, since the length of the circumference of the unit circle is $2\pi 1^2=2\pi$ units, it follows for each number x that

$$\sin(x+2\pi)=\sin x,\quad \cos(x+2\pi)=\cos x,$$

$$\sin(x+\pi)=-\sin x,\quad \cos(x+\pi)=-\cos x,$$

$$\tan(x+\pi)=\frac{\sin(x+\pi)}{\cos(x+\pi)}=\frac{-\sin x}{-\cos x}=\tan x$$

$$\text{for } x\neq\frac{\pi}{2}+m\pi,$$

and

$$\sin(\pi/2-x)=\cos x,\quad \cos(\pi/2-x)=\sin x.$$

It should be emphasized that in these typical relations x does not mean x radians and certainly $\pi/2$ is not 90°.

To obtain the usual formulation for $\cos(x_1-x_2)$, two arcs, x_1 and x_2, may be measured upon the unit circle in the proper manner from the point (1, 0). The other extremities of these two arcs are, by definition $(\cos x_1, \sin x_1)$ and $(\cos x_2, \sin x_2)$, respectively. By application of the Pythagorean Theorem, the square of the distance between these two points is

$$(\cos x_1-\cos x_2)^2+(\sin x_1-\sin x_2)^2.$$

If, now, a new rectangular axis system is introduced in such a manner that the new origin coincides with the old origin and the new reference axis passes through the point previously denoted by the point $(\cos x_2, \sin x_2)$, the two extremities previously studied have the coordinates $[\cos(x_1-x_2), \sin(x_1-x_2)]$ and (1, 0). The square of the distance between these two points may now be written:

$$[1-\cos(x_1-x_2)]^2+\sin^2(x_1-x_2).$$

After equating this value to the one previously found, only a few simplifications are required before obtaining the desired relation,

$$\cos(x_1-x_2)=\cos x_1\cos x_2+\sin x_1\sin x_2.$$

With just a little ingenuity, the reader can discover transformations upon this latter result to yield the usual formulas for $\cos(x_1+x_2)$, $\sin(x_1-x_2)$, and $\sin(x_1+x_2)$.

With such a start, the entire fundamental theory of trigonometry can be developed with a minimum of difficulty. The student will understand the properties of the trigonometric functions, and he will be prepared for the important applications of trigonometry in science and mathematics.

To consider the special application of trigonometry to angles, following the presentation of the subject as just outlined, it is desirable to construct a system of angular measure having the property that an angle at the center of a unit circle has the same measure as its intercepted arc. The radian system has the specified property. The trigonometric functions of such a central angle may be defined as identical with the functions of its arc. The classical numerical trigonometry of the triangle readily follows, but, as already indicated, this part of the course probably deserves minimum attention.

The authors of this article hope that more and more teachers of trigonometry will realize the importance of placing major emphasis upon trigonometric functions of numbers, not angles. Perhaps other readers of this journal will have additional suggestions for accomplishing this purpose.

Interscholastic Contests in Mathematics

Kenneth B. Morgan of Mount Kisco, New York (Box 576) is making a study of mathematics contests to determine their value in the modern educational program. He is investigating the value of both intra-class and interscholastic contests. He would appreciate receiving from those who have had experience in conducting these, their estimate of the value of such contests. The answers to such questions as the following would be helpful:

1. Do interscholastic contests give the good student a deeper interest in mathematics and a desire to do more individual work?
2. Do those winners of contests who continue mathematics in college make exceptional records in mathematics?
3. Are they a means of encouraging mathematics clubs to do worthwhile extra-curricular work?
4. Are intra class contests helpful in the creating of enthusiasm for the subject?
5. Are they helpful in classes of poor students as well as in the classes of the more able?
6. Do contests encourage dishonesty?
7. Do they tend to place emphasis on the impractical puzzle rather than on the practical problem?

Oversocialized Mathematics

By Francis Behn Riggs
Cambridge 38, Mass.

Teaching of mathematics in the public schools of this country appears to be based, to a considerable degree, on a belated reaction against earlier American individualism. Accordingly one of the aims of teaching mathematics is to develop thoughtful contributing citizens in a democratic society. To carry out this aim both the content and procedure have been socialized for many years—more recently through the inclusion of war-time material. There is much to be said for this point of view. By using everyday illustrations and problems of a democracy in action, interest and a consequent achievement have pervaded the school rooms, which could never have been reached by the old problem of *A* and *B* doing "a piece" of inscrutable work while *C* pays for it, and *D* is injected merely to produce the desired academic confusion.

There are, however, other aspects of the socialization of arithmetic which arise from a restricted philosophic view of just what this socialization may signify, for socialization is often circumscribed by a limited scientific point of view. It has been suggested that in order to fulfill the social requirements of the students the content of arithmetic should be based solely on the frequency of the material as it occurs in the various current trades, businesses and professions in the particular locality which the particular school serves. The same note was struck by the Progressive Education Association work shop at Bronxville in 1937 where the criteria for evaluating mathematics programs stressed personal living; individual, personal and social relationships; social-civic relations; economic relaitons. Of course other criteria are used in determining the content of the curricula, but the point of view as outlined continues to be so prevalent that it needs careful consideration.

Against such exclusively socio-scientific viewpoint, fruitful as it may be, the following objections are offered.

It we limit the content of the curriculum to the material which statistics show is useful to the average citizen, we get precious little arithmetic, which might be all right if it were not based on the fallacy that whatever exists in an average citizen's mind is tantamount to all that *might* profitably exist. The philosophy of the limited curriculum eliminates the private uses, the private hints to further more advanced mathematical or parallel thinking. The functional idea, the general case, the elementary concepts of the calculus, the solid geometry of perception—are all pregnant with possibilities for the individual growth of even many elementary students. To wait for the rare student who has shown unusual ability, who has expressed his desire to become an engineer or scientist, is unfair to average students, some of whom might profit by an exposure to such mathematical thought, if well taught.

If a student *is* interested in exclusively socialized mathematics, it is largely because of the attached values: sociological, economic, historic. Children are interested in arithmetic for many reasons. In a survey of 808 students made in 1937,[1] it was found that arithmetic was far more popular than any other subject. In fact, "Arithmetic, art, reading and physical education were the only subjects liked best more often than they were liked least," and of these subjects, and on the same basis, arithmetic was over three times as popular as its nearest competitor, art. Now the reasons, as given by the children, for their preference was 43% for the sense of accomplishment: "I can do it," "I get

[1] National Elementary School Principals, 16th Year Book, July, 1937, p. 341.

good grades," "I can do it better than I used to"; 43% because of liking the subject matter: "I like this study," "It's interesting," "It's fun," "I like it better than any other"; and only 14% because of its attached values. In other words, if the survey of 808 students is a fair sample of average children, it would appear that young pupils like arithmetic because of the intrinsic values it has for those who study it, rather than because of "values" ascribed to it by educators in the fields of sociology, economics and history.

The contention here made is that children's interest in arithmetic is timeless, is not wholly dependent on their immediate group needs, and requires fewer outside props to sustain that interest than is usually supposed. A committee[2] on arithmetic has written, "It is not enough to cast education in terms of children's present interests and desires alone. A child's life at the moment is not a thing in itself, nor is it all-sufficient and self-contained. His life is a continuum; his future is more real than his present." It is in the timeless continuum that a child's interest may reach out toward fields, forbidden perhaps, but not beyond his understanding, if accompanied by appropriate teaching in simple language. The Committee expressly states that "teaching on felt needs and interest only is inadequate," and suggests that "all the dynamics of learning" be used.

The socialization of arithmetic, supposedly the outcome of majority need, is in fact based on minority opinion.

A more utilitarian objection to over-socialization may be found in the transfer of pupils. I have seen a good many young boys who have come from a school system where they learned and could write why both the president and the garbage man needed arithmetic to make them effective citizens, but who, on the other hand, showed less ability in problem solving and manipulation than was reasonable to expect. When these boys were transferred to a school where problem solving and manipulation were practiced rigorously, the result was a social as well as a mathematical misfit.

Disproportionate socialization carries its effects right into college, and can place children at a serious disadvantage when they become University students.[3]

A clearer recognition of the philosophic assumptions underlying the content and method of our arithmetic programs is needed. Without speculation and *a priori* thinking we cannot hope to advance to new fields. The great educators such as Commenius, Pestalozzi, Froebel and Eliot made their contributions through what they thought might and *ought* to be, argued *a priori* and tested *a posteriori*. It is the function of philosophic thought to suggest, to make wide generalizations on frankly insufficient data. It is the function of the scientific method to test the validity of such suggestions. The philosophic point of view is really an inclusive scientific point of view and might result, for example, in not one mathematics curriculum for children of a given grade, but more tentative differentiated curricula. It might result in more work being attempted with the functional idea, the general case, intuitive solid geometry to precede plane geometry, methods of approximation and so on. It might result in the retention (for the minority) of those forms and concepts which may have no immediate group value but which may prove highly interesting and hence, in the broader sense, useful.

[2] National Society for the Study of Education, 29th Year Book, 1930, Report Committee on Arithmetic.

[3] Hassler, J. O. "What Price Enrichment?", THE MATHEMATICS TEACHER, October, 1941.

Pet Peeves of a Mathematics Teacher

By M. H. AHRENDT
Shortridge High School, Indianapolis 7, Indiana

THE dictionary defines "peeve" as a colloquial word meaning "complaint" or "fit of resentment." The word "pet" means "favorite" or "indulged." Thus, to use the more mild of these definitions, a "pet peeve" is a "favorite complaint." Now every person has a right to his favorite complaints, and in a democracy he has the right and sometimes the obligation to express them. The writer does not feel, however, using the stronger of these two definitions, that he has "indulged in fits of resentment" over the problems that he has met as a mathematics teacher. But he does feel that he is entitled to make three complaints, and that perhaps other teachers of mathematics will find common ground with him in some of them.

The first of these pet peeves we shall refer to as the "answer school" of mathematics teaching. Quite a few teachers and students are members of this group. The philosophy of the answer school is that only the answer counts. The method is not important. The job of mathematics is to get answers. The end justifies the means, and if by "hook or crook" we can get the right answer, then our mathematics is all right.

Now all we have to do is look at some of the work turned in by students who have absorbed the answer-school philosophy to see why some teachers list this type of procedure as a pet peeve. The student trained in the answer-school philosophy doesn't bother to make a drawing when the problem involves a geometric figure. He has little need for organization. No one needs to understand the work anyway. All we need is the answer. He frequently omits his equal signs. Or he puts them in the wrong places so that equations are run together. But that is all right. The form in which it is written doesn't matter as long as he gets the answer. If he finds at the end of the problem that a result must be multiplied by or added to something, he is likely to tack the computation onto the last equation, where it is as abnormal and unsightly as a tumor. This jumble of figures frequently does not make sense even to the student who wrote it. Ask him to explain it and you will find that his understanding is as hazy as his paper. But the answer is right, and that is supposed to redeem the whole process. Such mathematics leads to confusion and misunderstanding, but it is the kind that develops when only the answer is important.

A well written solution in mathematics begins, whenever it is appropriate to the problem, with a neat, properly labeled drawing. Unknown quantities are suitably represented by literal numbers, and the numbers so used are properly designated and explained. The formulas needed are quoted. Given quantities are stated and explained before they are substituted in the formulas. The method used is made clear. Wherever needed, explanatory matter in the form of phrases or sentences is used. The answer is interpreted. Sometimes the proper interpretation of the answer is the most important part of the problem. The development is logical and clear from beginning to end. If someone else with the requisite background reads the solution, he can understand it without having to work it. More important still, the student himself can understand it if he happens to reread his solution after a lapse of several days. Even in routine drill work, such as solving equations for practice, the student should write out enough to make the method clear, with the logical step-by-step development showing on the paper.

The difficulty is that the type of work done under the answer-school philosophy is inconsistent with the very nature of

mathematics. It makes mathematics a matter of solving puzzles. Now solving puzzles may be a highly entertaining activity for a while, but its results are not lasting. The student soon forgets because he never understood.

The second of our pet peeves we shall call the "trickster school" of mathematics teaching. This term is meant to include those persons who teach mathematics by means of tricks and short-cuts. The trickster school has caused an immense amount of misunderstanding about mathematics. Many persons are surprised at the seeming tricks that a mathematician can use, and some of the more alert ones are delighted. But good mathematicians don't in general use tricks. They use logic. Persons subject to the trickster-school philosophy learn a lot *about* mathematics, but they seldom learn what mathematics is. The question in solving a problem is not what tricks can we use, but what is the *sensible* thing to do.

Perhaps the most widely taught of all tricks is transposition. Even good students in high school fail to see any logic behind transposition. To them it is a convenient and more or less magical way of getting rid of any number that happens to be in the way when they solve an equation. When they get an equation of the type $ax=b$, someone is almost sure to suggest that they transpose the a. In an advanced algebra class recently the students factored the trinomial $2-3x+x^2$. Some of them got the factors $2-x$ and $1-x$. Others got the factors $x-2$ and $x-1$. Naturally the question was asked whether there was any way to show that the two answers were equivalent. It is interesting to note that most of those who had learned to solve equations by transposition thought that this trick was just what was needed to change the first pair of factors into the second pair. Why, said some, you can change the position of any number if you change its sign each time you move it. Fortuate is the high school student who has never heard the word "transpose."

It is possible to teach beginning algebra students to solve equations by determinants. They catch on to the trick quickly. But they completely miss the fundamental problem of eliminating one of the unknowns. When the same term appears in each member of an equation, we can "cancel" it out. It is just as quick and more reasonable to subtract the term from both sides of the equation. When the signs in an equation are inconvenient, we can automatically change the sign of each term. But is it just as efficient and more sensible to multiply both sides of the equation by minus one. In solving the equation $bx=k$, many a student will tell you to detach b from x and put it under k, and he is sure the rule is right because it produces the right answer. But the student who really understands mathematics will base his solution on the sensible idea that if equals are divided by equals, the quotients must be equal. To most students the idea that the form of a fraction may be changed by multiplying the numerator and denominator by the same number is just a rule. Not many students know why this procedure is valid.

The weakness in the trickster school is that it emphasizes tricks instead of fundamental understandings. Isolated tricks are soon forgotten. But understandings which in themselves seem reasonable are more likely to remain. The insidious thing about the trickster method is that it seems at the moment to produce great results. Students catch onto the tricks quickly and use them with facility as long as they remember them. It is easier to teach tricks than to teach the basic ideas of mathematics. It takes less time and we seem able to cover more ground. At the end of the year a class taught by the trickster method may actually outshine in performance a group taught by the other method. But the tricks prove only temporary while the fundamental understandings are more lasting. In giving students a foundation for mathematics, permanence is surely more desirable than a quick speed in manipulation.

In the solution of equations in particular, the student should be made to see the reasonableness of the methods used. When the student understands that to any pair of equal quantities, in mathematics or anywhere, certain obvious things may be done without destroying the equality, then his work begins to take on life and his conclusions to assume dependability. The place to teach the axioms is in beginning algebra.

There may be a few cases in elementary mathematics where it is hard to avoid a short-cut, as in the multiplication of signed numbers. Even in this case, however, the short-cut may be made reasonable by a suitable illustration even though it may not be possible logically to prove the rule. The teacher will need to help the student distinguish between tricks and conventions. There is no *reason* why plus and minus signs have to indicate opposites or a number above the line has to tell how many times the preceding quantity is used as a factor. We just agree to give these symbols a certain interpretation. But once the agreement is reached, the student must abide by the convention and must find the sensible way to apply it and to extend its use.

The time may arise when the advanced student needs some short-cuts to save time. If so, let him adopt them when the need arises and after his understanding of the fundamental principles is clear. The appropriate time for this hardly comes in high school.

Our third pet peeve we shall refer to as the "easiest-subject school." This group is intended to include those persons both within and without the field of mathematics who insist that mathematics is the easiest subject in the curriculum to teach. There is a surprisingly large number of such persons. Mathematics is either right or wrong, they argue. It is entirely objective. And this makes the papers so easy to grade. Well, if answers are all that count, then it is either right or wrong. But if the method and the development count, then the person may be mostly right and still get the wrong answer. Or he may get the right answer and still have much of the problem wrong. The careful teacher looks at more than the answer. Grading mathematics papers is much like grading papers in English composition. The English teacher doesn't look only at the last sentence. He looks for a clear and orderly development of the idea. And he checks the mechanical details such as punctuation and sentence structure. The mathematics teacher also looks for orderly development and correct mechanics, and it is as time-consuming to mark such errors in mathematics as it is in any other subject.

The members of the easiest-subject school maintain that mathematics is easy to teach because it is largely a matter of drill. If tricks and short-cuts are entirely adequate, then drill is the indicated procedure. But if fundamental understandings are important, then drill is only one of the easier aspects of the teaching problem.

It is also sometimes argued that mathematics is easy to teach because the subject matter of elementary mathematics does not change over a period of years. The subject matter of mathematics does not change in the sense that it becomes invalid or out of date. The theorems stated by Euclid two thousand years ago are as valid today as they were then. But the subject matter taught in any given classroom may well change from year to year. In no other high-school subject has the choice of topics to be taught been argued more intensely than in mathematics. Furthermore the methods of teaching many parts of mathematics are matters of disagreement. One has but to read a few of the articles in THE MATHEMATICS TEACHER to become aware of the strong controversy that surrounds many of the questions in our field. Mathematics is far from a static subject. The teacher of mathematics who has his procedures so well worked out that he can teach from year to year without change surely must not be fully alert.

This discussion points out some of the reasons why the members of the easiest-subject school so often get "under the skin" of mathematics teachers. The members of this school have usually also been influenced by the other two schools. In fairness to the majority of mathematics teachers it must be said that many of those who consider our field the easiest are those who have never worked in the field or have been in it only temporarily. Distant pastures, it is said, often look green.

If the mathematics teacher cannot lay claim to having the easiest field, then he can seek distinction in at least one other direction. There is no field that is of more fundamental importance in modern society. Without mathematics there could be no science, no industry, no business, perhaps even no politics. Without some mathematics not even the simplest organized society could exist. If all teaching of mathematics were completely discontinued, our present economy would certainly collapse within a generation. The job of the mathematics teacher is not easy. It is not free of problems. But it does not lack a challenge.

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On Square Root by Division

By E. W. Chittenden
University of Iowa, Iowa City, Iowa

In a recent article[1] Mr. William S. Tobey has written regarding the advantages of teaching root extraction by a method which he calls "Approximation and Division." I wish to suggest some improvements in the method he proposes and to discuss several related educational questions. Let us begin by considering the proposed modifications.

Let a denote a real number greater than 1 whose positive square root r is desired. In the method discussed by Mr. Tobey, an approximation x to r is found by interpolation, the quotient $y=a/x$ is found by long division, and the average of x and y provides a new approximation which is better than the first. Aside from the interpolation, this process was discussed in detail by Lewis E. Ward,[2] who established its convergence for any initial choice of a positive number x. He commented on the fact that in some cases the convergence is rapid. As a matter of fact, if x is near r, the error in the average of x and y is so small that it becomes desirable to replace the method of Mr. Tobey by the simpler method outlined below. We proceed to determine the error in the average of x and y, knowing the limits of error in the approximation of x.

Recalling that $r=a$, let e be the error in the approximation x, that is, $e=r-x$. Since $y=a/x=r^2/x$, it follows readily that the error $z-r$ in the derived approximation $(x+y)/2$, is given by the calculation

$$z-r=\frac{(x+r^2/x)}{2}-r=\frac{(x-r)^2}{2x}=\frac{e^2}{2x}.$$

Thus, assuming that x is greater than 1, whenever the error in x is less than 10^{-2}, the error in the next approximation will be less than one-half of 10^{-4}. That is, if x is correct to two decimal places, z will be correct to at least four.

We may take advantage of the rapid convergence of the process of division by carrying the division to twice as many places of decimals as have been obtained. It is not necessary to demonstrate the correctness of this rule. The student will soon realize its correctness through experience. Since the adjoined process of approximation increases the difficulty of teaching the method it can profitably be omitted. The following example compares the two methods.

It is desired to find the square root of 7 to three decimal places. By interpolation, Mr. Tobey obtains the approximation 2.6. By division he obtains 2.7, with the average 2.65. The next quotient and average are 2.64, 2.645, respectively. By division he obtains 2.647, and the average 2.646 is the answer given.

Compare the foregoing with the following computation. The number 3 is the nearest integer to the desired root. The average of 3 and 7/3 is 2.67. The result of dividing 7 by 2.67 is 2.622, and the average obtained is 2.646. Since the error in 2.646 is 0.00026 to five places, this number may be used as a divisor to obtain the root to six places, actually to seven. Thus, in this case the desired result is obtained with a single long division, while Mr. Tobey used three long divisions, found three averages, and required an interpolation.

In another illustration the square root of 1127 is found. The first approximation is 30. By division we obtain 37.67. The average of this with 30 is 33.8. Dividing again, we have 33.34, with the average 33.57. Since the error in this result is less than 0.002, the formula for the error in the

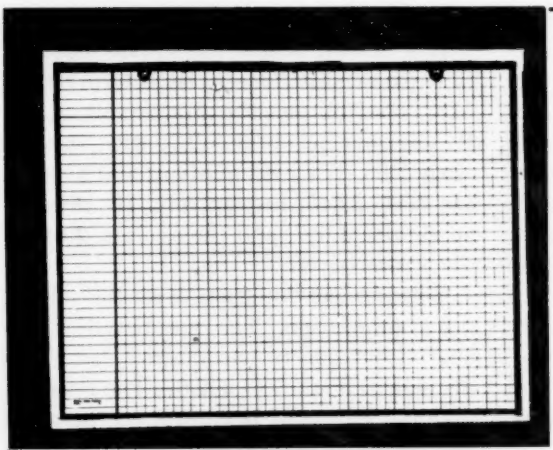
[1] The Mathematics Teacher, March, 1945, p. 131.

[2] On the computation of square roots. The Mathematics Teacher, 1931, pp. 86–88.

next approximation shows that it will be correct to six places of decimals at least. Similar figures are obtained by Mr. Tobey with less division but by a method difficult to explain and requiring about the same amount of computation.

I am doubtful of the value of teaching root extraction for its own sake. It is desirable for a student to understand the meaning of square root and the division process contributes most effectively to that end. It is my observation that unless it is desired to give practice in long division, the method in question is not sufficiently motivated. Most computers use tables of square roots, and it is my opinion that secondary school training is deficient to a marked degree with respect to the information given students regarding existing mathematical tables and their use. While I regard understanding of processes as an unavoidable essential of sound teaching, I deplore an expenditure of time in the mastery of methods of slight practical value, if any. Since long division is an important tool, it may be taught indirectly in higher grades through media such as the problem of root extraction.

In conclusion, I wish to urge strongly the systematic use in the schools of tables of squares and square roots along with such other tables as can be effectively employed without diversion of time from essential topics.



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The Retention of Arithmetic and Algebra in Relation to Achievement in Plane Geometry*

By MILDRED CHALLMAN
Monroe High School, Monroe, Michigan

INTRODUCTION

THE renewed interest in secondary school mathematics that resulted during preparedness and war periods did not instigate this investigation, as the problem here presented seemed important at any time in a plane geometry course. However, the criticisms from various sources of the content and methods of secondary school mathematics provided some motivation for this attempt to study one phase of the entire problem.

The starting point of the study was the receipt of a Navy arithmetic test that had been given to 3,230 navy recruits in 1940, with the resulting average scores made by them. The average score was low, but one could reasonably assume that the psychological factor of forgetting had operated for varying intervals of time. With this as a basis, one could study the problem of the relation between the retention and relearning of arithmetic and algebraic skills to pupils' achievement in plane geometry, where both are needed for satisfactory progress in the latter subject.

The purposes of this investigation were (a) to find how much and what parts of arithmetic and elementary algebra were recalled by high school pupils at the beginning and end of their study of plane geometry, (b) to determine whether improvement in these areas was evidenced at the conclusion of the geometry course, (c) to learn if achievement and retention in these same fields were related to achievement in plane geometry, (d) to find if achievement and general intelligence were related to any significant degree, (e) to see how high school geometry pupils compared with navy recruits on a test of arithmetic fundamentals, (f) to formulate some recommendations for the incorporation of the arithmetic and algebraic material necessary for the mastery of the numerical applications of geometry into the latter course of study, and (g) to try to explain the bases for some of the accusations that have been directed against the present practices and trends in the teaching and content of mathematics in the secondary schools.

A survey of the related literature from the first reported investigation by Thorndike in 1922 to the present time revealed no general conclusions on the problem of retention of either arithmetic or algebra.

THE METHOD OF INVESTIGATION

The data for this study were collected in plane geometry classes taught by the investigator in the public high schools of Fairmont, Minnesota, and Kankakee, Illinois, during the school years 1942–3 and 1943–4, respectively. Fairmont is located in southwestern Minnesota in a rich agricultural area, while Kankakee, near Chicago, is highly industrialized. Plane geometry was an elective subject in both schools, and the study includes 59 cases from Fairmont and 48 from Kankakee.

The Navy arithmetic test of twenty examples in fundamentals, and an algebra test of twenty simple problems in fundamental operations, equations and radicals, were administered to the geometry students in September and again at the close of school late in May.

The data were analyzed statistically by correlation methods and by tests of significance and reliability. Only those differences with probabilities that were at or below the one per cent level were con-

* Abstract of a Master's Thesis on file in the University of Minnesota Library, August 1945.

sidered significant. The statistical methods employed are not included in this report.

Presentation and Analysis of Data

The I.Q.'s of the Fairmont group were based upon the *Terman-McNemar Test of Mental Ability*, the mean I.Q. being 117.9 with a standard deviation of 9.16. For the Kankakee pupils the *Kuhlmann Anderson* tests, fifth edition, were used and the mean I.Q. was 115.7 with a standard deviation of 11.83. Because raw I.Q. scores based on different tests are not equivalent, the test of significance that showed no significant difference between the means of the two samples should be considered in this light.

The mean score made by the navy recruits on the twenty problem arithmetic test was 10.9. On the same test in September, the Fairmont group had a mean score of 14.9 and the Kankakee group 14.6. There was a significant difference between the means of the navy men and each of the high school aggregates, but this was not unexpected when one considers the lapse of time between attendance at school and enlistment in the navy, and the various and perhaps, limited arithmetic demands made upon these men during that period.

In May the means scores on the same arithmetic test were 16.4 and 16.8 for the two school groups. The standard deviations were 2.32 and 2.17, respectively. Again the two groups could be considered as random samples from the same population. The gain from September to May was significant.

The algebra picture was somewhat different. In September the means on this test of twenty examples were 8.4 and 9.8, with standard deviations of 4.35 and 4.17, respectively for Fairmont and Kankakee, and the two groups were homogeneous with respect to this characteristic. But in May the first group had a mean score of 13.7 with a standard deviation of 3.43, and the second group, 15.7 and 3.09. Only the means of the scores in May of the two school groups differed significantly. A possible explanation for the poorer showing on the part of the Fairmont students may be that they had had three or four different algebra teachers, due to changes in faculty personnel at the start of the war. In each group, however, there was a significant gain in algebraic skills from September to May.

Coefficients of correlation between I.Q.'s and scores on the arithmetic and algebra tests ranged from .18 to .97, those below .34 being unreliable. On this evidence one would not be justified in assuming that an I.Q. score based on one mental test would always be a very reliable criterion for guiding pupils into mathematics courses, or that it could be used consistently as a reliable means of estimating individual retention and ability to relearn this particular mathematical content.

The correlations between the test scores gave some indication that students who had remembered more algebra over the summer vacation had also acquired more during this particular school year and that there was a tendency for pupils to maintain absolutely the same rank on the arithmetic test as on the algebra test. There was also some evidence that, after the interval of three months, those students who retained absolutely more of the algebraic processes also retained relatively more of the same abilities they once possessed.

Correlations between final achievement in geometry and the various test scores ranged from .49 to .69, so one could conclude that ability in the performance of arithmetic and algebraic processes, as measured by these tests, was one factor associated with achievement in plane geometry.

Analysis of Errors

Chi-Square tests revealed that there was no significant difference between the distributions of errors made by the two school groups on the arithmetic test in

September; in May the difference was of doubtful significance. The Fairmont pupils showed significant improvement in solving a proportion and in extracting the square root of a number. In the Kankakee group the same types of errors seemed to persist over the nine-month period.

On the algebra tests there were significant differences on the kinds of errors made by the two groups. In September the Kankakee pupils surpassed those at Fairmont in the solution of equations and in operations with radicals, but the unit on the latter topic had been omitted from the Fairmont course. From September to May, both groups improved in their ability to handle operations with radicals—understandable perhaps, because in geometry, radicals are presented in more meaningful situations. A more detailed analysis of the errors showed no other definite pattern of improvement for the two school groups.

Educational Implications

At the outset of the course in plane geometry, an instructor in the subject should be aware of the factor of forgetting and not expect any great amount of proficiency in arithmetic and algebraic skills from his pupils. With some review and frequent applications of these processes, a significant amount of this knowledge can be relearned. Certain types of errors seem to persist, however, and it would be well to investigate the amount of knowledge and the most frequent types of errors made by pupils in arithmetic at the end of the eighth year, and in algebra at the end of the ninth year.

The superiority of tenth grade students over navy recruits in arithmetic skills would seem to substantiate the forgetting factor also, on the assumption that many of the latter group had probably not made use of these processes for varying intervals of time.

The correlations of I.Q. with grades in mathematics and with the results of the retention tests used in this study were not consistent with the practice of using the I.Q. alone as a means of guidance into, or prediction of success in, mathematics courses. Had the I.Q. scores been based on more than one mental test, however, it might have yielded a more reliable basis for judgment.

Admittedly there are many factors which contribute to a pupil's progress and achievement in plane geometry, but the results of this investigation showed that proficiency in arithmetic and algebra were associated with progress and achievement, even if the relationship cannot be established as a causal one. A useful study would be to determine which of these mathematical processes are most frequently used in the numerical and algebraic applications of plane geometry, and from these, to build review material that would be focused upon these particular needs. The skills most lacking among students at the beginning of the geometry course in this investigation were in the areas of solving a proportion, finding square root of a number, operations with fractions and decimals, changing cubic yards to cubic inches, squaring a binomial, solving a quadratic equation, and simplification of radical numbers.

The significant improvement in the handling of radical numbers during the study of plane geometry was due in part to an inadequate coverage of the unit in the ninth year, but their use in concrete situations might also make them more meaningful than in algebra where they are usually considered only in the abstract.

The results of this study would seem to indicate that a teacher is not entirely justified in claiming that pupils have not learned, or have not been taught, certain skills, until he or she has investigated what these students once knew about these processes. An acceptance of the pupils' present status, presentation of material in meaningful situations, re-teaching, and drill will, if the variability and individual differences of the group are considered, help to remedy the situation.

A Program for the Teaching of Mathematics in High Schools and Colleges

By JENNIE P. KORMES
Brooklyn College, Brooklyn 10, N. Y.

A GREAT deal has been said and written recently about the inadequacy of the curricula of our colleges and the need of coordinated methods of instruction to bring about a better understanding by the student of the interrelation of the doctrines taught and a better preparation for his progress in business, profession or further studies.

The ideas and suggestions contained in this article are those of a teacher of mathematics who, for many years, has endeavored to bring about the very objectives stated above, and are therefore confined to the teaching of mathematics and its relation to other sciences. The author believes, however, that in order to achieve a successful and efficient college program, it is necessary to attack the problem of the pre-college preparation, that is, the high school curriculum and educational methods.

It is generally admitted that the vast majority of students entering college lack a satisfactory preparation to enable them to embark upon the study of the subject at the college level. And this lack is due not to the insufficiency of the material taught at high school but on the contrary to the overcrowded curriculum. The students do not master the essentials, do not have a thorough understanding of the principles, lack the ability to review by themselves the material they should know and do not possess the sense of responsibility for the material they have been taught.

There are, in general, three classes of students in our high schools: one group which will enter business or other occupation, the second which will continue with some type of professional training and the third which will enter college. All three groups alike must therefore receive a basic training which will form the foundation of their entire future life and become an integral part of their personality. Whatever mathematics they are taught must be useful to them in their endeavors and it can be useful if, and only if, it is thoroughly understood and mastered.

As mentioned above, it is the opinion of the writer that the principal cause of the inadequate mathematical preparation in high schools is the overcrowded curriculum. It attempts to cover too much material with the result that the students' knowledge is hazy and overcrowded with formulae which are easily forgotten.

Some colleges and educators suggest special refresher and review courses for the students who enter college. This method is neither the answer nor the solution of the problem. It only lulls the college student into a false sense of security but fails to teach him self-reliance. It is uneconomical because it constitutes a waste of time, effort and money on covering the same material over again. Finally it is unfair to the other groups of students who do not enter college and who went to high school to obtain the minimum basic training.

The writer proposes that the high school curriculum be considerably reduced so that it is possible to give the students a solid foundation and a thorough understanding of the material taught. Just what the irreducible minimum curriculum should be would be decided by a Curriculum Committee suggested in a subsequent portion of this article.

What is apparently lost by the limitation of the material is more than offset by the extension of the depth of the study and thorough mastery and understanding. Whenever the class is above the average, the teacher can go beyond the minimum

material required, not by introducing additional doctrine but by going deeper into the various phases of the prescribed material.

Throughout the program, stress must be placed on developing in the student the sense of responsibility for knowing the material.

The teaching at the high school level should, however, be not only intensified but also coordinated with other basic science courses taught. This may be done either by means of special courses or seminars or else by means of extra-curricular activities.[1]

The student should acquire an ability to look things up and to know where to look them up when he needs them. This can be accomplished by assigning topics and selected readings.

Let us turn now to the problem of the college curriculum. There we have to deal with two classes of students: those who intend to major in science (mathematics included) and those who intend to major in other fields. For the second class, there should be established a general course to train the students primarily in mathematical methods which impart clear and logical thinking and keen analytical approach to problems. The course should serve also to bring out the cultural value of mathematical discipline as well as its application to various sciences.

For the science majors, it is of paramount importance that there be perfect timing and coordination between the mathematics courses and the subject matter of their major. It must be emphasized at this point that the writer does not recommend the subordination or adaptation of the mathematics courses to courses in other subjects. On the contrary, an independent and rigorous presentation of the subject in its purest form is advocated. There can be no harm, however, if the material be so arranged that the student requiring it for the understanding of work in other fields is ready and prepared when he needs it. Furthermore, such an arrangement is beneficial not only to the instructors of the various sciences but even to a higher degree to the teacher of mathematics. The student who has learned a certain mathematical theorem or method in a purely abstract fashion and who has the opportunity to utilize this knowledge in the field of his major interest acquires not only a deeper understanding of the theory but also a keener interest in mathematics.

A program of such magnitude and on such a scale cannot be attempted by a single department or college. It is therefore proposed that a curriculum committee be formed composed of representatives of high schools, colleges and graduate schools of universities. The committee should include not only teachers of mathematics but also teachers of other sciences because a similar need for integration and coordination exists in other fields of instruction.

The committee would have two main objectives:

1. The development of an integrated curriculum of studies for high schools and colleges. The high school curriculum would have to prepare the student adequately for his college studies and the college curriculum train him properly for advanced graduate work.

2. The coordination and arrangement of the subject matter in such a manner as to enhance and improve the teaching of science as well as of pure mathematics.

The committee should be a standing committee because this type of program is not a static one and cannot be solved once and for all but rather requires continuous changes and improvements to keep up with the times.

The committee could formulate a preliminary program and introduce it on an experimental basis. Its members would observe and examine the results of such an experimental program and hold conferences and hearings for teachers interested

[1] See "Coordination of Mathematics and Science through Student Activities"—Jennie P. Kormes, *National Mathematics Magazine*, November 1945.

in the furtherance of the project. It would encourage individual teachers, schools or colleges in experimental work on their own initiative and study the results thus obtained in order to determine the value and usefulness of these methods in relation to the general program and objectives. The committee would also examine and recommend textbooks as well as supplementary reading material at various levels. Where it finds a lack of proper textbooks, it would make recommendations to the teaching body asking for the preparation of the required textbook. The draft of such a textbook could be then used experimentally in several places by different teachers and in this manner any practical difficulties could be removed prior to its publication for general use.

Local conditions, the type of students and schools must be given due consideration. The New York City school system would be an excellent proving ground for a program of this kind. If it is found successful, there is no reason why it could not be extended to high schools and colleges throughout the country.

EXAMINATION IN EDUCATION 267G.
THE TEACHING OF GEOMETRY

In the following list of questions, mark those that are all true with a (+) sign, and those that are false or partly so with a (−) sign.

( ) 1. The word "geometry" originally meant "earth measure."

( ) 2. If we consider any definite object, aside from such matters as its structure and purpose, there are four questions of a geometric nature that can be asked concerning it.

( ) 3. Euclid did not prove that if in a triangle ABC we know that $a=b$ then $\angle A=\angle B$ because he thought anybody doubted it.

( ) 4. The purpose of the course in demonstrative geometry is to make clear to the pupil the meaning of demonstration, the meaning of mathematical precision, and the pleasure of discovering absolute truth.

( ) 5. It is the systematizing motive that controls the pupil's interest in the study of informal geometry.

( ) 6. It is now generally the practice to postulate the superposition theorems on congruence.

( ) 7. The subject of limits is now pretty generally omitted by most teachers of geometry.

( ) 8. If we wish to preserve the course in solid geometry, it can be done in at least three ways.

( ) 9. In the European schools the number of propositions studied in geometry have generally been larger than in this country.

( ) 10. Whereas the modern tendency is to omit all memorized definition in arithmetic and for the most part in algebra, the opposite is true in geometry.

( ) 11. Euclid's sequence of propositions in geometry were universally followed for 1000 years.

( ) 12. Sufficient drill cannot be given in formal geometric proofs in plane geometry, hence the need for at least a unit on solid geometry.

( ) 13. For those pupils who may leave school at the end of the ninth grade a unit of plane geometry is not of importance.

( ) 14. One of the most interesting opportunities for teaching plane and solid geometry simultaneously is found in associating the circle and the sphere.

( ) 15. Euclid's brilliant success in organizing into a formally deductive system the geometric treasures of his times has caused the reign of science in the modern sense to be so long deferred.

( ) 16. The trend now is to make some solid geometry, algebra, and trigonometry an integral part of the course in formal geometry.

( ) 17. The method of superposition used in the congruence theorems is in harmony with the larger aim of geometric instruction in the schools.

( ) 18. Teachers of plane geometry should not hesitate to make as many allusions as possible to related propositions in three dimensions.

( ) 19. Swenson advocated graphic methods of teaching congruence in geometry.

( ) 20. There is no basic theorem that can be proved only by the indirect method of proof.

( ) 21. Indirect reasoning is known also as *reductio ad absurdum.*

( ) 22. The method of exhaustion is not a special case of the indirect method of proof.

( ) 23. The law of converse applies to the proposition "If DC is the perpendicular bisector of the line AB, then all points on the bisector are not equally distance from A and B."

( ) 24. To be truly initiated into the spirit and meaning of logical processes, the pupil must acquire the mastery that comes only from analytic thinking.

( ) 25. The main thing in the analytic method of proof is the analysis which leads to a known true relationship.

( ) 26. If a plane figure is such that, if it be turned over about a certain one of its lines, the new position of the figure coincides with the original, it is said to be symmetric with respect to the line.

( ) 27. The doctrine of formal discipline does not assert that the formal training or the mental power gained from the study of certain school subjects carries over to all other activities.

( ) 28. The older doctrine of mental discipline was based on an erroneous "faculty psychology."

( ) 29. No significant results were obtained from Thorndike's study on "Mental Discipline in High School Subjects."

( ) 30. Transfer of a negative type has been found to occur principally among sensory-motor habits.

Write a few words to show that you understand the meaning of the following terms:

1. Congruent figures
2. Doctrine of identical elements
3. Inductive method
4. Converse of a theorem
5. Opposite of a theorem
6. Informal geometry
7. Model proof
8. Postulate
9. Devil's Coffin
10. Pons Asinorum
11. Law of cosines
12. Hero's formula
13. Pickett's list of theorems

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Models for Certain Pyramids

By JOSEPH A. NYBERG
Hyde Park High School, Chicago, Ill.

THE PROOF for the volume of a triangular pyramid involves showing that the pyramid is one of theee equal pyramids that form a prism. To make paper models of the three pyramids most pupils make the prism first and then measure certain lines in it to find the edges of the pyramids. But even when making the oblique prism the pupil has no direct geometric method for making the third lateral face after he has constructed two of the faces; that is, after making a pattern for $\triangle ABC$, for parallelogram $ABDF$, and for parallelogram $BCED$, he cannot construct parallelogram $ACEF$ because he knows neither the diagonal CF, nor the angles at the vertices, nor any altitude of the parallelogram.

The purpose of this article is to show a geometric method for finding CF, not in the expectation that any pupil will carry through the construction but so that teachers can do the work and thereby find some suitable lengths for the lines for pupils to use. My own experiments have led to the following choice of lines:

$AB=4$, $BC=3\frac{1}{2}$, $AC=4\frac{1}{2}$, $AD=4\frac{3}{4}$, $BD=3\frac{3}{4}$, $CD=4\frac{1}{4}$, $CF=5\frac{13}{16}$.

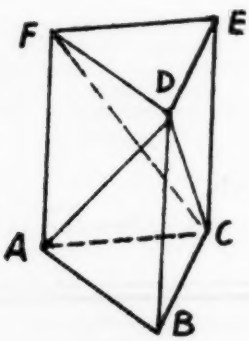

FIG. 1

If ABC is to be the base of a triangular pyramid, we construct on its sides the triangles ABD, BCD, and ACD, and then fold the triangles upward until the vertices marked D meet. As an exercise in plane geometry we note that the perpendiculars from D to AB in $\triangle ABC$, from D to BC in $\triangle BCD$, and from D to AC in $\triangle ACD$ are concurrent. This is a difficult exercise even for a bright pupil unless he draws three circles: one with A as a center and AD as a radius, one with B as a center and BD as a radius, and one with C as a center and CD as a radius. (Incidentally, the pupil *ought* to know that these circles would be useful in the proof since these are the circles that he draws when constructing the figure.) The perpendiculars to the sides of $\triangle ABC$ are then seen to be the common chords of three intersecting circles, and these can be proved concurrent by a familiar theorem about chords.

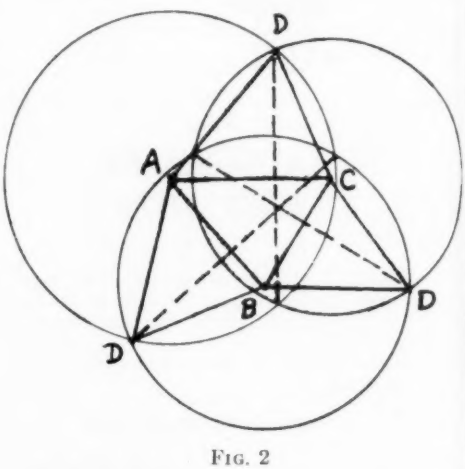

FIG. 2

In a solid geometry class the pupil would reason that when he folds up the triangle ABD to make the pyramid, the point D moves in a plane that is perpendicular to AB; likewise, point D of $\triangle BCD$ moves in a plane that is perpendicular to BC. If two intersecting planes are each perpendicular to a third plane, their intersection line is perpendicular to that plane,

and is therefore the altitude of the pyramid. Hence the altitude from D of the pyramid intersects the plane ABC in the point where the common chords intersect. Call that point O.

The altitude, h, of the pyramid can then be found geometrically. It is the arm of a right triangle of which the hypotenuse is the edge AD and one arm is AO. As a check, we can use the right triangle having BD for the hypotenuse and BO as an arm, or the triangle having CD for the hypotenuse and CO as the arm. Further, knowing now the altitude of the pyramid, we can find its volume if it is wanted.

Consider next the problem of constructing the three pyramids that form the oblique triangular prism $ABCDEF$.

For the first pyramid we select six lines, AB, BC, AC, AD, BD, and CD, and construct a paper pattern for the pyramid D–ABC.

For the pyramid $C-DEF$ we know FD, DE, EF, CD, and CE, since these lines equal respectively AB, BC, CA, CD, and BD; but the pattern cannot be constructed until CF is found. This is done as follows:

First construct $\triangle FDE$ congruent to ABC. Then with E as a center and CE as a radius draw a circle; and with D as a center and DC as a radius draw a circle. Then the foot of the altitude from C to the base FDE of the pyramid is on the common chord GH of the circles.

Next construct a right triangle (not shown in these figures) having CE for the hypotenuse and the altitude h of the pyramid $D-ABC$ as an arm. Let k be

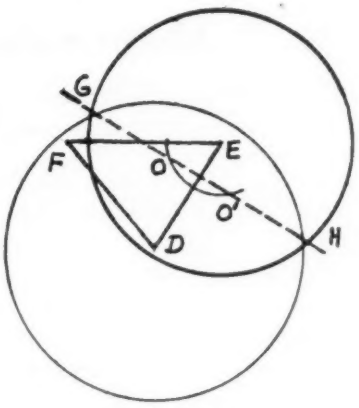

FIG. 3

the other arm of this right triangle; that is, $k=\sqrt{CE^2-h^2}$.

Then with E as a center and k as a radius draw an arc intersecting GH at points O and O'. Then either O or O' is the foot of the altitude from C of the pyramid $C-FED$.

Suppose O is the correct foot of the perpendicular. Then the desired line FC is the hypotenuse of a right triangle whose arms are h and FO.

To decide wheter O or O' is the correct foot of the perpendicular, it is necessary (1) to find the altitude of pyramid $C-ABD$ regarding C as the vertex, and then (2) find FC by repeating the above work for the pyramid $C-AFD$ starting with the $\triangle AFD$, finding the common chord of two intersecting circles and so forth. Again two values of FC are obtained, one of which agrees with a previous one. In most cases this work is unnecessary since the correct foot of the perpendicular can be selected from the appearances of the pyramids.

ON TO CLEVELAND!
FEBRUARY 22 AND 23

Mathematics Takes a Holiday

(Having lost their numbers, students find them again in the most unexpected places!)

By WENDALL W. HANER
St. Joseph High School, St. Joseph, Michigan

"IF I COULD spend one whole day at the side of every student, pointing out to him the importance of numbers in everyday living, what a difference it would make in student interest!"

At this wishing well every mathematics instructor interested in his subject and alive to the thrill of student response frequently finds himself. And if the wish were granted? Probably, in his enthusiasm, the teacher would greatly overdo the matter. And his adventure in omnipresence might end in his becoming decidedly unwelcome in many quarters, and acquiring a reputation for a one-track mind in others.

Perhaps there is a better way to achieve the same result. Why not use a system which will make every student an eager assistant in demonstrating the importance of numbers to every other student in the thrill of an exciting game?

The plan is No-Math Day. On this occasion no number work is done in classes. Stories or general discussions take its place. Mathematics is officially blacked out, slumbering and forgotten. But under the surface, in countless real-life situations, attention on numbers is at fever pitch, for traps are everywhere. One's best friends are plotting intrigue and preparing subtle snares. "Math production" is at its peak, for the No-Math Game is on!

Its rules are simple. No student is to reply to a question requiring a number answer unless he first says "Math"—thus indicating his recognition of a need for numbers. If he fails to say the magic word, he is caught and his blunder figuratively forfeits his scalp to adorn the belt of his questioner. Each pupil is instructed to catch as many others as possible and as often as he can.

Students are cautioned not to let a victim know when he is caught, but to lead him on and run up the score! One expert once clinched a championship when he found a friend reading in the library. Keeping attention on the book, he asked the unsuspecting reader the number of pages read, the number of books read that month, the time spent in reading each day, and many more questions—until he had collected seventeen unrecognized number answers in five minutes!

The time of day, locker combinations, distances, dimensions, personal statistics, dates, addresses, and a multitude of kindred items with number content are stressed and explored with a vigor and thoroughness which a teacher alone could never produce. And, in addition, every pupil has an exciting day making and avoiding clever mathematical questions and immensely enjoying the fun of numbers.

The next day the secrets are out. Classes in mathematics become hours of fun as open-mouthed pupils learn how they played into the hands of clever schemers—*and how many places they used numbers that they had never recognized as mathematical situations before.* Even the teacher finds new number uses, for usually one or more members of a class will arrive with his name on the victim list!

No-Math Day can very profitably become a special event to be observed in the early part of each school year. Students soon begin to hear about it, look forward to it, plan for it, discuss it with their parents and classmates,—and begin their study of mathematics with a new eagerness because of it.

◆ THE ART OF TEACHING ◆

A Practical Project in Geometry

By Fanny Springsteen
Stephenson, Mich.

Perhaps one effect of our Pre-induction courses of recent years will be the restoration of mathematics to its rightful place in the school curriculum. In the decade before the War we certainly swung to the far extreme from the spirit of Plato who, more than 2000 years ago founded his Academy in Athens and over the door placed this challenge, "Let None Ignorant of Geometry Enter Here."

As schools broke away from strictly College Entrance requirements and introduced Business, Agriculture, and Home Economics as courses which a pupil could pursue and earn the desired diploma, English and History were carried along in their respective and respected places, but mathematics and science were left in the lurch. Mathematics particularly came to be regarded as having importance only for the mental elite. Pupils steered around algebra and geometry as being "too hard," and advisors often encouraged this attitude.

This avoidance of these subjects deprived many "just average" pupils of both value and happiness. For there is value in setting up equations and in reasoning from the known to the unknown; and there is pleasure in meeting the challenge of solving correctly a difficult problem and of proving a baffling proposition.

In Stephenson Township High School, a rural Agricultural high school with an enrollment of 270 pupils in grades 9 through 12, an effort has been made to make courses in mathematics adequate to meet the needs of future college students and also to attract those pupils who might profit by and enjoy the practical side of the subject. The result has been that, while the courses are entirely elective, about 60% of the pupils elect algebra and about 50% enroll in plane geometry.

It is the purpose of this article to tell briefly of the plan used by the plane geometry class during the past year to help capitalize on this pupil interest. The project, supplementing the regular work, grew out of a class discussion regarding some of the illustrations the author had included in the text to show the wide use of rectangles in building, of circles in machinery, etc. The final outcome was the idea that each member of the class should choose a vocation or a field of activity and see what relation it bore to the principles set forth in Geometry. No attempt was made to avoid duplication of choices, the only suggestion being that they have interest for the pupil making the choice. Also a choice need not be final if one of greater interest was discovered after the investigation was begun.

Naturally several boys immediately chose Geometry in Aviation, and the material to which the search led in books and magazines was almost unlimited. A boy whose brother was in the Navy picked Geometry in Navigation, and another whose father owned a garage found many applications of Geometry in Machinery. But it was a girl who took Geometry in Carpentry and showed its value in making a picture-frame, a top for a dressing-table, a dormer window, a pig-pen, and a roof for the barn.

Two girls chose Geometry in Architecture. One correlated with her course in Ancient and Medieval History, while the other centered on more modern buildings

ranging from a hip-roofed barn to the sky-scrapers of the big cities.

Three pupils selected Geometry in Music and turned in results as different as the pupils themselves. One dwelt entirely on the staff and symbols used in writing music, while the other two gave emphasis to the forms and designs of a great variety of musical instruments.

A girl whose ambition is to study Medicine produced almost a masterpiece on Geometry in Medicine and Surgery. Her pen-and-ink drawings of instruments and machines were little short of works of art. The War maps led one girl into the realm of Cartography. A boy who had difficulty in settling on a subject finally became very interested in Geometry in Advertising. Another, after some hesitation, decided on railroads and was enthusiastic over the construction of tracks, switches, and signals, and made some excellent drawings of the same.

Geometry in Designing and in Interior Decorating found favor with several girls who were interested in Home Economics. Lovely books were made, pointing out the application of the geometric principles in wall-paper, linoleum, jewelry, clothing, and in construction and arrangement of furniture. Their pages of patterns for quilt blocks would have delighted their grandmothers. Geometry in 4-H Work covered the various enterprises sponsored by that organization and included diagrams of the grounds and buildings of the Summer Camp.

Perhaps one of special interest was Geometry in Sports, the work of a girl who was one of the school Cheerleaders. Drawings to scale of fields for baseball and football, and courts for tennis, basketball, and other games were carefully constructed. Illustrations cut from magazines and catalogs emphasized the fact that most games are played with geometrical objects.

Geometry in Farming, Building, Engineering of various types, Landscaping, and City-planning were other topics that received attention. In all investigations were made in the several fields by 30 pupils. Of the results turned in, only two seemed to bear the earmarks of required work. All others went far beyond any standard requirements that might have been set up. Many comments indicated genuine pupil interest of a high degree.

Perhaps a word should still be said regarding the form in which the reports were finally submitted. This, too, was optional with the individual pupil, and so they varied all the way from a couple of pages of rather carelessly written material to excellent note-books of 40 or 50 pages of beautifully arranged illustrations and descriptions and having appropriately designed covers. More than half were of the latter type and the big majority were such as will be proudly displayed at our annual exhibit of school work. Some were written up as well constructed themes, giving the illustration subordinate place; others made the illustration the center, using brief statements to point out the geometrical relationships.

One never knows what the trend of a new class will be. Right now I am wondering if this year's group will develop a similar interest. If so, we might give some good reports in class with the use of a new opaque projector just purchased for our school. In this way the pupils' findings in a field could more easily be shared with the whole class and each one's knowledge be broadened.

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IN OTHER PERIODICALS

By NATHAN LAZAR

Midwood High School, Brooklyn 10, New York

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2. Coolidge, J. L., "Two Dimensions," pp. 557–562.
3. Rosenbloom, P. C., "An Elementary Constructive Proof of the Fundamental Theorem of Algebra," pp. 562–570.
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5. Funkenbusch, William, "Hyperspacial Figurate Progressions," pp. 571–573.
6. Recent Publications, pp. 574–577.
7. Problems and Solutions, pp. 578–589.
8. News and Notices, pp. 589–591.
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   (c) Oldenburger, Rufus, "Mexican Mathematics."
   (d) Marine Corps Criticizes the Teachers of Mathematics.
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2. Rogers, Mary C., "Membership Message," pp. 6–7.
3. Grossnickle, Foster E., "Needed Changes in the Teaching of Mathematics," pp. 9–15.
4. Karst, Otto J., "Development of the Type Forms of the Equations of the Conic Sections from the Eccentricity Definition," pp. 15–17.
5. Fehr, Howard F., "Reconversion and Reconstruction" (Editorial), pp. 17–18.
6. Webb, Harrison E., "Andrew S. Hegeman: a Tribute," pp. 18–19.
7. "Book Reviews," pp. 20–21.
8. "News and Notices," pp. 22–23.
9. "Problem Page," p. 24.

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2. Fisher, Genevieve Herriott, "A Mathematics Assembly Program," pp. 5–7.
3. Betz, William, "The Redirection of Education in the Light of War Experience," pp. 7–13.
4. Ulmer, Gilbert, "A Story of Number Symbols for High School Students," pp. 13–15.

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2. Sleight, Norma, "Conics Are Fun," pp. 787–791.
3. Breslich, E. R., "Mastery of the Fundamentals of High School Mathematics: A Graduation Requirement," pp. 807–818.
4. McFarlan, L. H., "A Direct Proof for the Addition Formula for Tangents," pp. 837–838.
5. Block, William E., "Magic Squares and Cubes—An Analysis of Their Structure, Methods of Construction and a Complete Set of Figures for a $7\times7\times7$ Magic Cube," pp. 839–850.
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7. Shuck, G. R., "A Complex Quantity Slide Rule," pp. 862–864.
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Program of the Annual Meeting of the National Council of Teachers of Mathematics

Cleveland, Ohio

Hotel Statler, Euclid Avenue and Twelfth Street

February 22 and 23, 1946

THURSDAY, FEBRUARY 21

8:00 P.M. Meeting of the Board of Directors of the National Council of Teachers of Mathematics

FRIDAY, FEBRUARY 22

9:30–11:30 A.M.

Euclid Room—Mathematics in the Elementary School

Leader: Edith Woolsey, Sanford Junior High School, Minneapolis, Minnesota

1. The Fundamentals with Meaning
   Elizabeth Kerr, Heminger School, Akron, Ohio
2. Dr. J. T. Johnson, Chicago Teachers College
   The Case for Decimal Versus Common Fractions
3. Devices used to put meaning into the Arithmetic of the Primary Grades
   Margaret West, Almira Elementary School, Cleveland, Ohio
4. The Review Versus the Telescoped Reteaching of the Work of Preceding Grades
   R. L. Morton, Ohio University
5. Discussion

Pine Room—Mathematics for Grades Ten to Fourteen

Leader: H. W. Charlesworth, East High School, Denver, Colorado

1. Problems in Administering a Program for Functional Competence
   George E. Hawkins, Lyons Township High School and Junior College, LaGrange, Illinois
2. Adjusting Teaching to Requirements of General Education
   Walter H. Carnahan, Technical Extension Division, Purdue University, Lafayette, Indiana
3. Needed Reforms in the Content, Organization and Methods of Teaching Mathematics in Grades Ten to Fourteen
   Dr. C. N. Shuster, State Teachers College, Trenton, New Jersey
4. Discussion

1:30–2:50 P.M.

Euclid Room—Mathematics in the Junior High School

Leader: Florence Brooks Miller, Shaker Heights Junior High School, Shaker Heights, Ohio

1. Methods used in the selection of pupils for the Study of Algebra and Geometry in the Cleveland Schools.
   Ona Kraft, Collinwood High School, Cleveland, Ohio
2. Popular Mathematics or Socialized Arithmetic for Non-Academic Students
   Anthony Marino, Head of Mathematics Department, East High School, Youngstown, Ohio
3. The Junior High School Textbook—Plus
   Veryl Schult, Supervisor of Mathematics, Washington, D. C.
4. Reckoning on the Line
   E. W. Schreiber, Western Illinois State Teachers College, Macomb
5. Discussion

Pine Room—The Training of Teachers of Mathematics

Leader: L. H. Whitcraft, Ball State Teachers College, Muncie, Indiana

1. Needed Reforms in Arithmetic Teaching and Their Meaning for Teacher Training Institutions
   Robert L. Morton, Ohio University, Athens, Ohio
2. Equipping Mathematics Majors to Teach in the High Schools of the Future
   Ruth W. Stokes, Winthrop College, Rock Hill, South Carolina
3. Some Proposals Relating to the Preparation for Teaching Mathematics
   E. R. Breslich, University of Chicago
4. Discussion

3:00–4:15 P.M.

Euclid Room—Informal Report of the Work of the Commission on Postwar Plans

Leader: Raleigh Schorling, Head of the Department of Mathematics, University High School, Ann Arbor, Michigan
Panel-Members of the Commission

6:30 P.M.

Discussion Banquet—Euclid Room—Hotel Statler

Presiding: President F. Lynwood Wren, George Peabody College for Teachers, Nashville, Tennessee
Subject of Discussion: Report of the Commission on Post War Plans
Address: William E. Wickenden, President of Case School of Applied Science, Cleveland, Ohio

SATURDAY FORENOON, FEBRUARY 23

Cleveland Board of Education Auditorium, Rockwell Avenue and Sixth Street

8:30 A.M.

Business Meeting

9:15 A.M.

Multi-Sensory Aids
Leader: E. H. C. Hildebrandt, Northwestern University, Evanston, Illinois

1. The Use of Models in the Teaching of Plane Geometry
   Frances M. Burns, Oneida High School, Oneida, New York
2. Mother Nature is a Mathematician
   Walter H. Carnahan, Assistant Professor of Education, Purdue University, Lafayette, Indiana
3. Preview of Films—Property Taxation
   Encyclopedia Britannica Films Inc.
4. What Arithmetic Can Films Teach?
   John H. Lewis, Encyclopedia Britannica Films Inc.
5. Some Army Methods and Aids Useful for the Teaching of Mathematics
   Lt. Henry W. Wyer, Signal Corps Photographic Laboratory, Army War College, Washington, D. C.

11:15 A.M.

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EDWIN W. SCHREIBER, *Secretary*
Western Illinois State Teachers College, Macomb, Illinois

OFFICIAL BALLOT FOR THE YEAR 1946

Mark the ballot below immediately, enclose it in an envelope and mail it to the above address. It must reach Macomb by Feb. 19, 1946.

INTRODUCTION OF CANDIDATES

Mrs. Elsie Parker Johnson—Head of Math. Dept., Oak Park-River Forest Twp. High School, Oak Park, Ill.; Past President Womens Math. Club of Chicago; former Director of the National Council.

Mr. Carl N. Shuster—Head of Math. Dept., State Teachers College, Trenton, N. J.; Director of the Council; Director Ass'n. of Math. Teachers of New Jersey.

Mr. Wm. A. Brownell—Prof. Educ. Psych., Duke Univ., Durham, N. C., formerly member of the National Arithmetic Committee of the Council.

Mr. H. W. Charlesworth—Head Math. Dept., East High School, Denver, Colo.; Director of the Council for six years.

Mr. Lee E. Boyer—Member Math. Faculty, State Teachers College, Millersville, Pa.; a Contributor to the 17th. Yearbook.

Mr. Walter H. Carnahan—Purdue Univ., Lafayette, Ind. He was President of Cent. Ass'n. of Sci. and Math. Teachers in 1945.

Mr. Houston T. Karnes—Louisiana State Univ., Baton Rouge, La. Chairman. La.-Miss. Branch of National Council in 1940.

Miss Ona Kraft—Teacher of Mathematics in Collinwood High School, Cleveland, Ohio.

Miss Edith L. Mossman—Teacher of Mathematics in Berkeley High School, Berkeley, Calif.

Miss Veryl Schult—Supervisor of Mathematics, Div. 1–9, Washington, D. C. Director of the Council for three years.

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

OFFICIAL BALLOT FOR THE YEAR 1946

For President, 1946–47 (Vote for one)

( ) Elsie Parker Johnson ( ) Carl N. Shuster ( ) _____

For Second Vice-President, 1946–47 (Vote for one)

( ) Wm. A. Brownell ( ) H. W. Charlesworth ( ) _____

For the Board of Directors, 1946–48 (Vote for three)

( ) Lee E. Boyer ( ) Walter H. Carnahan ( ) Houston T. Karnes

( ) Ona Kraft ( ) Edith L. Mossman ( ) Veryl Schult

( ) _____

Name and address of voter_____

(Necessary to check against membership list)